MATHEMATICAL PROGRAMMING AND ITS APPLICATION:

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IN
STATISTICS

BY
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UNDER THE SUPERVISION OF
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Certified that Mrs. NUJHAT JAHAN, has carried out research on "*Mathematical Programming And its Application*" under my supervision and that her work is suitable for submission for the degree of Master of Philosophy in Statistics.

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Nujhat Jahan

CONTENTS

PREFACE

This manuscript intend to present some literature on "Mathematical Programming and its Application". The dissertation consist of four chapters with a comprehensive list of references at the end.

Mathematical Programming is concerned with the determination of a minimum or a maximum of a function of several variables, which are required to satisfy a number of constraints. Such solutions are sought in diverse fields, including engineering, operations research, management science, numerical analysis and economics etc.

Chapter - I, gives an introduction of the subject. It deals with the basic knowledge of Mathematical Programming and its advancement. It also contains some practical examples of the numerous applications of Mathematical Programming.

In chapter-II, the construction of block design is viewed as a Mathematical Programming problem and its solution is also discussed.

In chapter-III, the problem of manpower scheduling is discussed as a Network Flow problem. An algorithm with example is also discussed.

The optimum integration of survey is introduced in chapter-IV, with some of the selection schemes. Mathematical formulation of Des Raj for Integration of Surveys has also been discussed as a Mathematical Programming problem.

CHAPTER - I

INTRODUCTION

1.1 AN OVERVIEW :

According to the "Journal of Operations Research Society, U.K.," Operations Research is defined as - " OR is the application of the modern methods of mathematical science to complex problems arising from the direction and management of large systems of man, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of system, incorporating measurements of factors such as chance and risk with which to predict and compare of outcomes of alternative decisions, strategies or controls. The purpose is to help management determine its policy and direction scientifically."

Optimization and uncertainty are dominant themes in operations research. It is the application of mathematical analysis to managerial problems. The operations researcher makes a contribution to this problem solving effort by mathematical techniques to obtain a solution. These optimization methods are collectively known as mathematical programming.

Mathematical programming is a termed coined by Robert Dorfman around 1950, and now is a generic term encompassing linear programming (LP), integer programme (IP), convex programming, nonlinear programming (NLP), network flow theory, dynamic programming under uncertainty etc. etc.

Programming problems in general may either belong to the deterministic class or probabilistic class. By deterministic class it is meant that if certain actions are taken then it can be predicted with certainity that what will be (a) the requirements to carry out the actions and (b) the outcome of any actions. Programs involving uncertainty are called the probabilistic. The outcomes of a given action may depend on some chance event such as the weather, traffic delays, government policy, employment levels, or the rise and fall of customer demand. Sometimes the distribution of the chance event is known, sometimes it is unknown or partially known. In some cases uncertainty arises because of the actions of competitions.

1.2 MATHEMATICAL FORMULATION :

A general mathematical programming problem can be stated as,

$$\left.\begin{array}{lll} & \text{Minimize } f(X) & \\ \text{Subject to } & g_j(X) \geq 0 & j = 1, \ldots\ldots m \\ & X \geq 0 & i = 1 \ldots\ldots n \end{array}\right] \quad \ldots (1.2.1)$$

Where $X' = (x_1, x_2 \ldots\ldots\ldots x_n)$ is the vector of unknown decision variables and $f(x)$, $g_i(x)$ are the real valued functions of the n real variables $x_1 \ldots\ldots x_n$.

The function $f(x)$ is called the objective function, and the inequalities $g_j(x) \geq 0$ are refereed to as the constraints. A maximization problem can always be converted into a minimization problem by using the identity :

$$\max f(X) = - \min (-f(X))$$

That is, the maximization of $f(X)$ is equivalent to the minimization of $\{-f(X)\}$. The functions $f(X)$ and $g_j(X)$ are assumed to be continuous or continuously differentiable. This assumption is necessary to make the problem more tractable to theoretical treatments.

The mathematical programming problem stated in (1.2.1) is also known as a constrained optimization problem. An optimization problem without any constraint is called unconstrained optimization problem.

1.3 MATHEMATICAL PROGRAMMING METHODOLOGY :

PROBLEM FORMULATION :

Problem formulation begins by detailed observation of the real world system. The researcher attempts to specify the objectives of the study. This usually involves careful collection of data for variables, objective function, parameters and constraints in problem setting.

MODEL CONSTRUCTION :

Model construction means converting verbal statements into mathematical expressions. The mathematical expressions must accurately depict the real system they are modelling. The generated model must express the operation of the real world system as accurately as the real system itself. In this step we carefully define the precise meaning and relationship of the model variables, objective function, parameters and constraints.

MODEL TESTING AND ANALYSIS :

After a model is constructed, its accuracy and reliability must be verified. This is called model testing. Once we are sure of the model's accuracy and reliability, we can use the model to generate a solution to the problem in the real-world system. The solution procedure for mathematical programming problems employ an algorithm. Different modeled problem situations require different algorithms.

INTERPRETATION, EVALUATION AND IMPLEMENTATION OF THE MODEL'S RESULT :

Having obtained solution or solutions one has to interpret the solution to understand, if it is indeed a solution. Some algorithms generate solutions with such computational complexity that only experienced mathematical programmers can interpret the solution. Understanding of the algorithm used is essential in obtaining an interpreting solution. Once interpreted, the decision maker must evaluate the feasibility of the solution. Some real-world restrictions or constraints on possible solutions cannot be mathematically expressed in a model, but must still be considered if a solution is to be selected. The evaluation of a solution permits the screening of any feasible solutions.

1.4 ADVANCEMENTS IN MATHEMATICAL PROGRAMMING TECHNIQUES :

Since the end of world war II, mathematical programming has developed rapidly as a new field of study dealing with applications of the scientific method to business operations and management decision making. But we can trace the existence of optimization methods to the days of Newton, Lagrange and Cauchy. The differential calculus methods of optimization was introduced by Newton and Leibnitz. The foundation of calculus of variations was laid by Bernoulli, Euler, Lagrange and Weirstress. Lagrange introduced his famous Lagrange Multiplier Technique to solve the constrained optimization problems. Cauchy made the first application of the Steepest Descent method to solve unconstrained minimization problems. In spite of these early contributions, very little progress was made until the middle of the twentieth century.

Linear programming was developed in 1947 by George B. Dantzig, Marshall wood, and their associates, as a tool for finding optimal solutions to military planning problems for the United States Air Force. The early applications were primarily limited to problems involving military operations, such as military logistics problems, military transportation problems, procurement problems, and other related fields. The numerical procedure for solving a linear programming problem introduced by Dantzig is known as Simplex Method. But the method was not available until it was published in the Cowles Commission Monograph No. 13 in 1951.

Kuhn H.W. and TUCKER A.W. (1951) published an important paper, "Nonlinear programming", dealing with necessary and sufficient conditions for optimal solutions to mathematical programming problems, which laid to the foundations for a great deal of later work in nonlinear programming.

Charnes A. and Lamke C. (1954) published an approximation method of treating problems with separable objective function subject to linear constraints. Later the technique was generalized by Miller to include separable constraints. In 1955, a number of papers by different authors dealing with the quadratic programming began to appear. Beale (1959) gave a method for quadratic programming. In the same year Wolfe P. transformed the quadratic programming problem into an equivalent linear programming problem, using K.T. conditions, which could be solved by Simplex method. The other authors who gave techniques for solving quadratic programming problem were Houthakher H.S. (1960), Lemke C.E. (1962), Panne and Whinston (1964)., Graves R.L.(1967), Alloin G. (1970) and several others.

Interest in integer solution to linear programming problems arose early in the development of the field. One of the first papers to be concerned with the subject was that published by

Dantzig, Fulkerson and Johnson in 1954. Markowitz (1957) and Manne discussed numerical techniques and some nonlinear programming problems which could be solved by integer linear programming. Ralph E. Gomory was the first to set forth a systematic computational technique for solving an integer linear programming problem for which it could be proved that convergence would be obtained in a finite number of iterations. This was done in 1958 for all integer case and in 1960 for the mixed integer case. A.H. Land and A.G. Doig (1960) gave a method which is especially appropriate for mixed integer programming. E.L. Lawler and D.E. Wood in 1966 applied the Branch and Bound technique of Land and Doig to various non-linear programming problems like the traveling salesman problem, the quadratic assignment problem. Wei Xuan Xu in (1981) gave a new bounding technique for the quadratic assignment problem.

Richard Bellman, made the major original contribution to the development of the dynamic programming technique and published his result in about 100 papers throughout the 1950's. A summary of this work is contained in his book "Dynamic Programming" published in 1957, and in the book, "Applied Dynamic Programming,", Co-authored with S.Dreyfus and published in 1962. Dynamic programming problems paved the way for development of the methods of constrained optimization. The contributions of Zoutendijk (1966) and Rosen to nonlinear programming during the early part of the 1960's have been very significant. Although no single technique has been found to be universely applicable for nonlinear programming problems. The work by Carrol (1961) and Fiacco and McCormick (1968) made many a difficult problem to be solved by using the well known techniques of unconstrained optimization. R Hooke and T.A. Jeeves gave a direct search method in 1961 for unconstrained optimization. M.J.D. Powell (1964) gave an efficient method for finding the minimum of a function of several variables without calculating derivatives. The other authors who made contribution for unconstrained optimization are H.H. Rosenbrock (1960), R. Fletcher and C.M. Reeves (1964), W.C. Davidon (1968), R. Fletcher (1970) gave a new approach to variable metric algorithms. L. Grandinetti (1982) gave an updating formula for quasi-newton minimization algorithm.

Geometric programming was developed in the 1960's by R.J.Duffin, E. Peterson and C.Zener. Geometric programming provides a systematic method for formulating and solving the class of optimization problems that tend to appear mainly in engineering designs. D.S.Ermer (1971) used geometric programming for optimizing the constrained machinery economics problem. Works are still going on geometric programming and its sensitivity analysis. R.S.Dembo (1982) applied sensitivity analysis in geometric programming.

G.B. Dantzig (1955), A Charnes and W.W.Cooper (1959) developed stochastic programming techniques and solved problems by assuming design parameters to be independent and normally distributed. The basic idea used in solving any stochastic programming problem is to convert the stochastic problem into an equivalent deterministic problem. The desire to optimize more than one objective or goal while satisfying the physical limitations led to the development of multiobjective *programming methods. Goal programming is a well known* technique for solving specific types of multiobjective optimization problems. The goal programming was originally proposed for linear problems by Charnes and Cooper (1961).

Developments for new techniques for solving mathematical programming are still going on. Kachian (1979) gave a polynomial algorithm for linear programming. N.Karmarkar (1984) gave an excellent method for solving linear programming problem. His method is named as New Polynomial-Time algorithm for linear programming. A recent contribution to integer programming was due to Saltzman and Hiller (1991).

1.5. MATHEMATICAL BACKGROUND :

VECTOR : An n component vector x is a row or a column array of n numbers given by

$$X' = (X_1 \ldots\ldots X_n) \qquad or \qquad X = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

EUCLIDEAN SPACE : An n-dimensional Euclidean space denoted by E^n is the collection of all n component vectors.

HYPERPLANE : The set of points $X \epsilon E^n$ satisfying, $C_1X_1 + C_2X_2 + \ldots\ldots + C_nX_n = Z$ (not all $C_i = 0$) (1.5.1) is called a hyperplane for given values of C_i and Z.

The hyperplanes corresponding to different values of Z are parallel.

LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

A set of vectors, $a_1, a_2 \ldots . a_m$ from E^n is said to be linearly independent if there exist scalars λ_i not all zero such that

$$\lambda_1 a_1 + \lambda_2 a_2 + \ldots\ldots + \lambda_m a_m = 0 \ldots\ldots (1.5.1)$$

If the only set of λ_i for which (1.5.1) holds is

$$\lambda_1 = \lambda_2 = \ldots\ldots\ldots \lambda_m = 0$$

then the vectors are said to be linearly independent.

BASIS : A basis of E^n is a linearly independent subset of vectors from E^n which spans the entire space.

The set of unit vectors from E^n, that are (1, 0 .... 0), (0, 1............0) ....(0, 0...........1) form a basis of E^n. Basis for E^n is not unique.

MATRICES : An mxn matrix is a rectangular array of mxn numbers, called the elements, in m rows and n columns. Matrices are usually denoted by capital letters such as A, B, C and their elements are denoted by small letters such as a, b, c etc.

Symbolically :

$$A = ((a_{ij})) = \begin{pmatrix} a_{11} & a_{12} & ..a_{1n} \\ a_{21} & a_{22} & ..a_{2n} \\ & \vdots & \\ a_{m1} & a_{m2} & ..a_{mn} \end{pmatrix}$$

The element in row i and column j is denoted by a_{ij}.

MATRIX INVERSE : Given a square non-singular matrix A i.e. $|A| = 0$, there exist another square matrix B satisfying the relation.

$AB = I$ ....(1.5.2)

then B is the inverse of A.

TRACE : The sum of the diagonal elements is called the trace of the matrix. thus :

Trace A = $\sum_{i=1}^{n} a_{ii}$

EIGEN VALUE : If for a given matrix A of nth order, we find a number λ and a vector X such that

$$AX=\lambda X$$

is satisfied, then λ is said to be an eigen value or characteristic value and X an eigen vector of the matrix A.

CONVEX AND CONCAVE FUNCTIONS : An n-variable real valued function $f(X)$ is said to be convex if for any two distinct points X_1 and X_2.

$$\{\lambda X_1 + (1-\lambda)\ X_2\} \le \lambda f(X_1) + (1-\lambda) f(X_2) \ldots\ldots (1.5.3)$$

A function $f(X)$ is said to be concave if $-f(X)$ is convex.

A function $f(X)$ is strictly convex or concave if strict inequality holds in (1.5.3). A linear function may be treated as convex as well as concave.

QUASI CONVEX AND CONCAVE FUNCTIONS : A function $f(X)$ define on E^n is said to be quasi convex if for any two point X_1, $X_2 \in E^n$.

$$f[\lambda X_1 + (1-\lambda)\ X_2] \le \max\ [f(X_1),\ f(X_2)] \ldots\ldots (1.5.4)$$

We call a function quasi-concave if $-f(X)$ is quasi convex.

PSEUDO CONVEX AND CONCAVE FUNCTIONS : A differentiable function $f(X)$ is pseudo convex if.

$$\begin{aligned} &\nabla f'(X)\ (Y-X) \geq 0 \\ &f(Y) \geq f(X) \end{aligned} \qquad (1.5.5)$$

If $-f(X)$ is pseudo convex then the function is said to be pseudo concave.

BASIC SOLUTION : Consider a system AX=b of m simultaneous independent linear equations n unknowns, where A is an mxn matrix of rank m (<n). A solution obtained by setting any (n-m) variables equal to zero, and solving the resulting system is called a basic solution to the given system of equations.

The m variables which are not sets equal to zero are called basic variables. The remailing (n-m) variables which are set equal to zero are called non-basic variables.

CONVEX SET : A set X in E^n is called a convex set if given any two points X_1 and X_2 in X, then,

$$\lambda X_1 + (1-\lambda)\ X_2 \in X \ for\ each\ \lambda \in [0,1]$$

EXTREMENT POINT : A point X is called an extreme point of convex set if and only if there do not exist other points X_1, X_2, $X_1 \neq X_2$, in the set such that,

$$X = \lambda X_1 + (1-\lambda) X_2 \qquad 0<\lambda<1$$

ADJACENT EXTREMENT POINT : Two distinct extreme points X_1, X_2 of a convex set X are called adjacent if the line segment joining them is an edge of the convex set.

GLOBAL MINIMA : A global minimum of the function f(X) is said to be attained at X_0 if,

$$f(X_0) \le f(X) \text{ for all } X$$

LOCAL MINIMA : A local minimum $f(X_0)$ of the function f(X) is said to be attained at X_0 if there exists an $\epsilon > 0$ such that $f(X_0) \le f(X)$ for all X, satisfying $|X_0 - X| \le \epsilon$

FEASIBLE SOLUTION : A vector X which satisfies the constraints and the non-negativity restrictions of the mathematical programming problem is called its feasible solution.

OPTIMUM SOLUTION : Any feasible solution which optimizes the objective function of a mathematical programming problem is called an optimum solution.

6 BRANCHES OF MATHEMATICAL PROGRAMMING :

The mathematical programming techniques are to find the minimum or maximum of a function of several variables under a prescribed set of constraints. Thus depending on the nature of these functions and the restrictions on the decision variables, mathematical programming problem can be broadly classified into two categories.

a) Linear Programming Problem (LPP)

b) Nonlinear Programming Problem (NLPP)

5.1 LINEAR PROGRAMMING :

A linear programming problem is a problem of minimizing or maximizing a linear function in the presence of linear constraints. An LP problem may stated as :

$$\left.\begin{array}{ll} \text{Minimize } Z = c_1x_1 + c_2x_2 + \ldots\ldots\ldots + c_nx_n & \\ \text{Subject to,} \quad a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n \geq b_1 & \\ \qquad\qquad a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n \geq b_2 & \\ \qquad\qquad \vdots & \\ \qquad\qquad a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n \geq b_m & \end{array}\right] \quad (1.6.1)$$

$$x_1,\ x_2 \ldots\ldots\ x_n \geq 0$$

$$Z = C_1x_1 + \ldots\ldots + C_nx_n = \sum_{j=1}^{n} C_jx_j$$

is the objective function. The coefficients C_1, C_2 ...... C_n are the known cost coefficient and X_1, X_2 ...... X_n are the decision variables to be determined. The inequality

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$$

denotes the i^{th} constraint. The coefficient a_{ij} , i = 1, ...m. j = 1, 2 ...... n are called the technological coefficients. These technological coefficients form the constraint matrix A are given below :

$$A = \begin{pmatrix} a_{11} \cdots\cdots\cdots a_{1n} \\ a_{21} \cdots\cdots\cdots a_{2n} \\ \vdots \\ a_{m1} \cdots\cdots\cdots a_{mn} \end{pmatrix}$$

The column vector whose i^{th} component is b_i is referred to as the requirement vector. The restrictions $x_j \geq 0$ are the nonnegativity restrictions.

The constraint $\Sigma\ a_{ij}.x_j \geq b_i$ can always be converted to an equation by subtracting nonnegative variables X_{sj} from the LHS of the constraint. Thus the constraint becomes,

$$\Sigma\ a_{ij}.x_j - x_{sj} = b_i$$

In this case the variables X_{sj} is known as slack variable.

If a variable x_j is unrestricted in sign, then it can be replaced by $x_j' - x_j''$ where $x_j' \geq 0$ and $x_j'' \geq 0$. i.e. $x_j = x_j' - x_j''$.

An LP problem can be stated more conveniently using matrix notation.

Minimize $\quad Z = CX$

Subject to $\quad AX = b$ ....(1.6.2)

$\quad X \geq 0$

ASSUMPTIONS OF LINEAR PROGRAMMING

The solution procedure of linear programming problem is based on a number of assumptions. Linear programming may not provide a desired optimal solution when the underlying assumptions are removed from the problem.

1. LINEARITY & ADDITIVITY :

   The primary assumption of linear programming is linearity in the objective function and in the side constraints. This implies that the measure of effectiveness and utilization of each resource must be directly and precisely proportional to the level of each individual activity. Also the activities must be additive. The total measure of effectiveness must be the sum of the measure of effectiveness of each individual activity. Joint interactions are impossible for the amount of resources used in some of the activities.

2. DIVISIBILITY :

   Linear programming presupposes the complete divisibility of the resources utilized and the units of output produced. That is, it is assumed that the decision variables can take on fractional values. Resources and activities are considered to be continuous within a relevant range.

3. FINITENESS :

   The need for optimal decision making arises from a relatively scarcity of productive resources. The problem in linear programming is the optimal allocation of limited resources to alternative activities to achieve a specific objective. There must be a finite number of resource restrictions and available alternative activities. If the decision maker is faced with an unlimited number of alternative activities, linear programming can no longer be used for finding an optimal solution.

4. CERTAINITY AND STATIC TIME PERIOD :

It is assumed that the coefficients of the decision variables in linear programming are known with certainity; all the coefficients such as unit profit contribution, prices, and the amount of resources required per unit of output are known constants. The available resources must also be known accurately. This assumption is reasonable if the variance of the input output coefficients is not significant; i.e. unit profit contribution and resource usage per unit of output do not fluctuate widely. Thus linear programming implicitely assumes static time period.

PROPERTIES OF LP PROBLEM :

Linear programming problems possess certain specific properties which make these problems easier to solve as compared to nonlinear programming problems. These properties are :

(i) The set of feasible solutions which satisfy the constraints and the non-negativity restriction is a convex set having a finite number extreme points (corners).

(ii) The set of all vectors $\underline{x}' = (x_1, x_2 \ldots\ldots x_n)$ which yield a specified value of the objective function is a hyperplane. Furthermore the hyperplanes corresponding to different values of the objective function are parallel.

(iii) If the optimal value of the objective function is bounded, at least one of the extreme point of the convex set will be an optimal solution and starting at any extreme point it is possible to reach an optimal extreme point in a series of steps such that at each step one moves along to an adjacent extreme point.

(iv) A given extreme point is optimal if and only if the value of the objective function at that extreme point is atleast as great as the value of the objective function at each adjacent extreme points.

In nonlinear programming problem some or all of these features may be absent creating difficulties in solving them.

1.6.2 NONLINEAR PROGRAMMING :

Nonlinear programming emerges as an increasingly important tool in economic studies and in operation research. Nonlinear programming problems arise in various disciplines as engineering, business administration, physical sciences and in mathematics, or in any other area where decisions must be taken in some complex situation that can be represented by a mathematical model :

$$\left.\begin{array}{l} \text{Minimize } f(X) \\ \text{Subject to } gi(X) \geq 0 \quad i = 1 \ldots\ldots m \\ \qquad\qquad X \geq 0 \end{array}\right] \quad ..(1.6.2)$$

The function f(X) or gi(X) or both may be nonlinear functions in X.

Interest in nonlinear programming problems developed simultaneously with the growing interest in linear programming. In the absence of general algorithms for nonlinear programming problems, it lies near at hand to explore the possibilities of approximate solution by linearization. The nonlinear functions of a mathematical programming problem were replaced by piecewise linear functions, these approximations may be expressed in such a way that the whole problem is turned into linear programming.

Kuhn & Tucker (1951) published an important paper "Nonlinear Programming", dealing with necessary and sufficient conditions for optimal solutions to programming problems, which laid the foundations for a great deal of later work in nonlinear programming.

If the nonlinear programming problem composed of differential objective function and equality constraints, then the optimization may be done by the use of Lagrange multipliers. A necessary condition for a function f(X) subject to the constraints gi(X) = 0, i = 1 ...... m to have a relative minimum at a point X^* is that the first partial derivatives of the Lagrange function defined as

$$L = f(X) + \sum_{i=1}^{m} \lambda_i g_i(X)$$

with respect to each of its arguments must be zero.

$$i.e. \ \frac{\delta f}{\delta X_j} + \sum \lambda_i \frac{\delta g_i}{\delta X_j} = 0$$

The above condition is also sufficient if the quadratic form,

$$Q = \sum \sum \frac{\delta^2 L}{\delta X_i \delta X_j} dX_i dX_j$$

evaluated at $X = X^*$ is positive definite for all values of dX for which the constraints are satisfied.

Nonlinear programming problems are in general difficult to solve. The Kuhn-Tucker conditions form the basis of many algorithms for nonlinear programming problems.

KUHN-TUCKER NECESSARY AND SUFFICIENT CONDITION

Let X^* be an optimal solution of the nonlinear programming problem,

$$\text{Maximize } f(X)$$

$$\text{Subject to} \quad gi\ (X) \geq 0 \quad i = 1, 2, \ldots\ldots m$$

$$X \geq 0$$

where all functions are differentiable and $X \in E^n$. Let us assume that the constraint qualification holds. Then there exist a vector $u^* = (u_1, u_2 \ldots\ldots u_m)'$ such that the following hold --

$$\left.\begin{array}{ll} \nabla_x \phi\ (X^*\ u^*) & \leq 0 \\ X^* \nabla x\ \phi\ (X^*\ u^*) & = 0 \\ \quad X^* \geq 0 & \\ \nabla_u \phi\ (X^*, u^*) & \geq 0 \\ u_x\ \nabla_u\ \phi\ (X^*, u^*) & = 0 \\ \quad u \geq 0 & \end{array}\right] \qquad (1.6.2)$$

Where the scaler function $\phi(x,u)$ is given by,

$$\phi\ (x,u) = f(X) + \Sigma\ U_i g_i(X) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.6.3)$$

The above six conditions in (1.6.2) are called the K-T necessary conditions.

These conditions are not sufficient. However under the following conditions the K-T conditions are sufficient also.

If in the NLPP (1.6.1) all functions are differentiable, f(X) is pseudo concave and gi(X), i=1, 2.....m are quasi concave and X^* satisfies the K-T necessary conditions then X^* is optimal for nonlinear programming problem.

The nonlinear programming problem can be classified into various categories. Some of them are,

i) Geometric programming

ii) Quadratic programming

iii) Integer programming

iv) Stochastic programming

v) Separable programming

vi) Multiobjective programming

vii) Fractional programming

viii) Goal programming etc.

Note that all these above categories are not mutually exclusive.

i) **GEOMETRIC PROGRAMMING :**

A geometric programming problem is one in which the objective function and the constraints are expressed as posynomial in X. It can be stated as,

$$\text{Minimize } f(X) = \sum_{i=1}^{N} C_i \prod_{j=1}^{n} X_j^{p_{ij}}, \quad c_i \succ 0 \quad x_j \succ 0$$

$$\text{Subject to } g_j(X) = \sum a_{ij} \prod_{k=1}^{n} X_k^{q_{ik}} \preceq 0$$

ii) **QUADRATIC PROGRAMMING :**

A quadratic programming problem is a nonlinear programming problem with an objective function which is a sum of linear and quadratic forms and linear constraints. That is,

Minimize $f(X) = C'X + X'DX$

Subject to $AX = b$

$X \geq 0$

Where $X' = (x_1, \ldots\ldots\ldots x_n)$
$C' = (c_1, c_2 \ldots\ldots\ldots c_n)$
$b' = (b_1, \ldots\ldots\ldots b_m)$

D is an nxn symmetric matrix, A is an mxn matrix.

iii) **INTEGER PROGRAMMING :**

If some or all of the design variables of a mathematical programming problem are restricted to take only integer values, the problem is called an integer programming problem.

A linear integer programming problem can be stated as,

Minimize $f(X)$

Subject to $g_j(X) \geq bi \qquad i = 1 \ldots\ldots m$

$X \geq 0 \qquad j = 1 \ldots\ldots n$

Xs' are integer

If either $f(X)$ or $gj(X)$ are nonlinear then the problem becomes a nonlinear integer programming problem.

iv) **STOCHASTIC PROGRAMMING :**

A stochastic programming problem is a mathematical programming problem in which some or all of the parameter or design variables are random variables. A stochastic linear programming problem can be stated as:

$$\text{Minimize } f(X) = C'X = \sum_{j=1}^{n} c_j x_j$$

Subject to $A.X \geq b$ $\quad i = 1 \ldots\ldots m$

or $\Sigma\, aij\, Xj \geq bi$ $\quad j = 1 \ldots\ldots n$

where some or all of the C_j, a_{ij} and b_i are random variables.

v) **SEPARABLE PROGRAMMING :**

A function f(X) of n variables is said to be separable if it can be expressed as the sum of n single variable functions $f_1(x)$, $f_2(X)$, ... $f_n(X)$, that is,

$$f(X) = \sum_{i=1}^{n} f_i(x_i)$$

A separable programming problem is one in which the objective function and the constraints are separable.

vi) **MULTIOBJECTIVE PROGRAMMING :**

The simultaneous constrained optimization of more than one objective functions is termed as multiobjective programming. it can be stated as, Minimize $f_1(X)$, $f_2(X)$ ...... $f_k(X)$

Subject to gj (X) ≤ 0 $\quad j = 1, 2 \ldots\ldots m$.

where $X' = (x_1 \ldots\ldots\ldots x_n)$ and K denotes the number of objective function to be minimized. Any or all of the functions $f_i(X)$ and $g_j(X)$ may be nonlinear. The multiobjective programming problem is also known as a vector minimization problem.

vii) **FRACTIONAL PROGRAMMING :**

Fractional programming problems form a special class of nonlinear programming problems. A fractional programming problem can be stated as,

$$Minimize\ Z = \frac{f(X)}{g(X)}$$

Subject to $h_k\ (X) \geq b_k \quad k = 1 \ldots\ldots m$

This is known as single-ratio fractional program.

A special case is the linear fractional programming problem, defined as :

$$Minimize\ Z = \frac{c'x + \alpha}{d'x + \beta}$$

Subject to $AX \geq b$

$X \geq 0$ where $d'X + \beta > 0$ assumed.

viii) **GOAL PROGRAMMING :**

The goal programming technique handles decision situations involving a single goal or multiple goals. The goal programming problem can be formulated as follows. Let $X'=(x_1 \ldots\ldots x_n)$ be the vector of decision variables. The vector X is required to satisfy certain environmental constraints. Let $K(CE^n)$ be the set of all points X satisfying the environmental constraints. Further, let gi $(i=1 \ldots m)$ be the desired value of the i^{th} goal and let $f_i(x)$ represent the actual result obtained for the i^{th} goal. With each goal is associated a functional which imposes a penalty for deviations from the desired goal, given by,

$$d_i(X) = | f_i(X) - g_i | \quad i = 1 \ldots\ldots m$$

In goal programming, a suitable function of the individual goal functionals is minimized subject to the environmental constraints.

1.7 APPLICATIONS OF MATHEMATICAL PROGRAMMING:

Mathematical programming models are widely used to solve a variety of military, economic industrial social etc problems. A 1976 survey in America to determine the use of mathematical programming in American companies shows that 74% of them use mathematical programming techniques to solve their various problems. In the following few practical situations are described which may be formulated and solved as problems of mathematical programming.

1. <u>RESOURCE ALLOCATION PROBLEM:</u>

An investor wishes to invest money in several available investment choices. The returns on investment in each of these choices are known and is expected to persist in the future. The investor wishes to diversify his investment fund. Then he has a problem of allocating his funds among these choices to realize the maximum return. This is an allocation problem and can be formulated as a linear programming problem.

2. <u>PRODUCT MIX PROBLEM:</u>

A company makes n kind of products for which m basic raw materials are used. There are certain restrictions on the availability of raw materials and demand of the products in the market. The company wants to know how much of each product should be manufactured to maximize its total profit. This problem can be formulated as a mathematical programming problem to get the optimal solution.

3. <u>BLENDING PROBLEM:</u>

Blending problem refers to situations where a number of components are mixed together to yield one or more products. There are restrictions on the available quantities of raw materials, restrictions on the quality of the products, and on the quantities of the products to be produced. The problem is, how to carry out the blending operation such that the profit is maximized.

4. <u>DIET PROBLEM:</u>

The list of possible foods that can be included in a balanced diet is available along with their nutrient contents and the cost per unit. The problem is to find a minimum cost balanced diet.

5. <u>TRANSPORTATION PROBLEM:</u>

Suppose there are m origins which contain various amounts of commodity that has to be allocated to n destinations. Suppose there are some restrictions on the availability and demand of the commodity in the origin and destination respectively. Let the cost of shipping a unit quantity from the origin i to the destination j be known. The problem is to determine the number of units to be shipped from the origin i to the destination j so that the requirements are satisfied and the transportation cost is minimized.

6. <u>ASSIGNMENT PROBLEM:</u>

Let there be n workers or machine for performing n jobs, one worker/machine can be performed only one job. Also, assume that the workers/machines vary in their respective capability for and suitability to a particular job. The assignment problem is to find the best way of assigning the n workers/machines to the n jobs.

7. <u>TRAVELLING - SALESMAN PROBLEM:</u>

A salesman starts from a given city and visits a group of cities. The problem is to find the shortest route for the salesman.

8. <u>CATERER'S PROBLEM:</u>

A caterer is faced with the problem of providing napkins for dinners on each of n consecutive days. The number of napkins required on the ith day is known. These requirements may be met by purchasing new napkins or by laundering napkins soiled at an earlier dinner at lesser cost. The problem is to meet the requirements for napkins at minimum cost.

9. <u>PRODUCTION SCHEDULING PROBLEM:</u>

A manufacturer has to produce several items. The cost of producing one item on regular time and on overtime are known. The variation of cost with time and also the capacity restrictions might make it more economical to produce in advance of the period when the items are actually needed and store them for future use at some storage cost. The problem is to determine the production schedule which minimizes the sum of production and storage costs.

10. <u>TRIM - LOSS PROBLEM:</u>

Paper mills produce rolls of a given, standard width. Customers require rolls of various width and hence, the rolls of standard width must be cut. In general, some waste occurs at the end of the cutting process, i.e trim loss. The manufacturer wishes to cut the rolls as ordered by his customers and to minimize the total trim loss. This application applies to similar manufacturing situations in which a standard roll, sheet etc must be cut with resulting trim loss.

11. WAREHOUSE PROBLEM:

Given a warehouse with fixed capacity and initial stock of a certain product, which is subject to known seasonal price and cost variations, and given a delay between the purchasing and the receiving of the product. The problem is to find the pattern of purchasing, storage and sales which maximizes the profit over a given period of time.

12. CRITICAL - PATH PLANNING AND SCHEDULING:

A characteristic of many projects is that all work must be performed in some well defined order, e.g. in construction work, forms must be built before concrete can be poured. This formulation concerns the scheduling of the jobs which combine to make a project. The problem is to select the least costly schedule for desired and feasible project completion time.

1.8 SOME SPECIALIZED APPLICATIONS OF MATHEMATICAL PROGRAMMING IN VARIOUS FIELDS:

The examples given in section 1.7 are simple practical problems that arise in the work of business people, managers of industry, and other executive whose main concern is with routine operational matters. In addition to helping with the solution of problems from that sphere of activity mathematical programming also finds numerious more abstract applications in engineering design, and scientific research. Few examples from those areas of application are given below.

1. <u>CURUE FITTING:</u>

Suppose that in some scientific research, say in biology or physics, a certain phenomenon $R(t) = at^2+bt+c$, is measured in the laboratory as a function of time t. The parameters a, b, and c are unknown and are to be estimated from the experimental measurements R. The problem is to find parameters a, b, and c so as to minimize the absolute value of the largest discrepancy between the measured values and corresponding theoretical values. The mathematical formulation of the problem is,

$$\text{Minimize } [\text{ maximize } |D_i| \text{ }]$$

where Di is the discrepancy between measured and theoretical values. Because we are minimizing the maximum absolute deviation, such a formulation is commonly called MINIMAX problem.

2. <u>NURSE SCHEDULING:</u>

A hospital administrator is in charge of scheduling nurses to work in different shifts. The number of nurses required during the day is known. The problem is to schedule the nurses to meet the requirements and to do so with the minimum number of nurses. This problem can be formulated as a linear programming problem and the desired optimal solution can be obtained.

3. CONSTRUCTION OF DESIGN:

In the theory of experimental design the choice of both the design and the model influence the conclusions drawn from the experiment. Thus problem of optimally choosing the design and the model may be formulated as a problem of mathematical programming. The model formulation of experimental design involves unknown parameters. These parameters can be estimated by using mathematical programming.

This problem is discussed in detail in chapter two of this manuscript.

4. LOCATION PROBLEM:

The location of a supply center to serve m customers having at fixed destinations in a city is to be selected. The commodity to be supplied from the center may be electricity, water, milk etc. The criterion for selecting the location of the supply can be formulated as a mathematical programming so as to minimize some distance function from the center to the destinations.

5. CLUSTER ANALYSIS:

Cluster analysis has been employed as an effective tool in scientific enquiry. One of its most useful roles is to generate hypothesis about category structure. The objective of cluster analysis is to assign n objects to k mutually exclusive groups while minimizing some measure of dissimilarity among the items. Mathematical programming techniques are applied to these problems for minimizing these dissimilarities.

6. QUALITY CONTROL:

The determination of an inspection plan for a continuous production system may be set up as a bicriteria problem. One may be interested in minimizing the expected unit cost of inspection and replacement. The unit cost included the cost of inspecting an item during production, the cost of replacing a defective item during inspection, and the cost of replacing defective items when returned by customers (or the next production line). The other objective may be to minimize the average outgoing percentages of defectives. Both of these objectives are nonlinear functions of decision variables. Zoints and Wallenius (1976) gave an iterative programming method for solving such problem.

7. ELECTRICITY SUPPLY SYSTEM:

Kwun & Baughman (1991) presents a model that integrates the supply planning of potential cogenerating industries with that of the electric utility suppliers. The model represents the technical alternatives for producing thermal energy in the industrial sector and electricity utility sector. The objective of the problem is to find the supply mix that minimizes the total cost of supply while meeting the required thermal electricity demands in a given system.

8. SCHEDULING OF FREIGHT TRAINS:

Freight trains over single-line track with crossing loops needs the determination of where and when such trains should cross. The dynamic rescheduling system schedules future train movements to minimize the overall cost due to late running of trains, and energy consumption. Nonlinear programming and discrete network methods may be used to provide the train controller with a continually updated crossing schedule.

9. CONSTRUCTION OF CYCLE ROSTERS OF WORKFORSE:

In creating a cyclic master roster for workforce is a high need of the recent years. Several authors e.g. Burns and Carter (1985), Koop (1988), Panton (1991) discussed the problem of determining the workforce size, given certain quality and recreational constraints on the structure of the roster.

10. INTEGER PROGRAMMING FOR ELECTRIC UTILITY CAPACITY PLANNING:

Mathematical programming techniques have been proposed for electric utility capacity expansion planning problems as early as the work of Masse and Gibrat (1957), Linear and non linear programming models of deterministic type problems were used. Anderson (1972) proposed more accurate probabilistic models for determining least cost investments in electric supply. Dynamic programming approach was also used. Sherali, Staschus, Haucuz (1987) used integer programming for an electric utility capacity planning problem.

11. OPTIMUM ALLOCATION IN STRATIFIED SAMPLE SURVEY:

The purpose of sampling theory is to develop the most economic procedures for sample selection and to obtain estimates of required precision. Sample survey problems can be formulated as optimization problems. Let n be the total sample size of stratified sample. The problem is to allocate the sample sizes n_1, n_2 .. n_h to various state such that.

$$\sum_{h=1}^{1} n_h = m$$

The problem is to find the integer values of n_h by minimizing the variance for fixed cost or by minimizing the cost for fixed variance. Dynamic programming technique may be used to solve this problem.

12. OPTIMUM ALLOCATION OF SURVEYS:

When two or more sample surveys are conducted on the same set of units it may be stipulated that the sample units are to have different probabilities for the different surveys. If we fix the sample size n to be same for each of the surveys, a sample drawn for one survey may not in general satisfy the restrictions on the probability of selection of units for the other survey. Thus we may have to draw different samples for different surveys and increased total cost. Hence we wish to reduce the cost of surveys by drawing the maximum possible number of common units for both surveys without violating the probability restrictions. This problem can be formulated as a mathematical programming problem.

This problem is discussed in some detail in chapter four of this manuscript.

13. <u>EMERGENCY SERVICE SYSTEM</u> :

In recent years mathematical programming models have been used to determine the number of units on emergency service. Urban emergency service systems have two distinct features -(i) probabilistic demands and service requirements over time, and (ii) probabilistic distribution of incidents and response units over the space of the city. The first feature is naturally analyzed by queuing models and the second feature gives rise to a travel time models. In addition simulation and dynamic programming can be used to analyze both features simultaneously.

14. <u>EDUCATION SYSTEM:</u>

Mathematical programming techniques have many contributions to make toward analyzing various operations carried out by educational organizations. There are many programming and scheduling problems having to do with the most efficient use of teachers and of teaching facilities. Schools are replete with queres at registration desks, gymnasium facilities etc. The routing of the school buses is not an insignificant task.

15. ENGINEERING:

Mathematical programming can be applied to solve many engineering problem. Typical applications of engineering disciplines are -

(i) Design of aircraft and aerospace structures for minimum weight.

(ii) Finding the optimal trajectories of space vehicles.

iii) Design of civil engineering structures.

(iv) Minimum weight design of structures for earthquake, wind and other types of random loading.

(v) Optimum design of electrical network, etc.

16. TRAFFIC SIGNAL SYNCHRONIZATION:

Traffic system and its modernization is a vital need in the recent urban life. Though traffic signals prevent chaos at busy intersections, but nobody likes the frequent stops that often occur on driving down a street with many signals. The number of stops can be held down by proper synchronization of the signals. The traffic signal synchronization was done by using mixed-integer linear programming by Little (1966).

17. PROJECT SELECTION:

The purpose of a project selection is to assist the administrator in prioritizing and funding the available projects. Three general categories of models have been developed, (i) checklists, (ii) economic indexes and (iii) portfolio models. Check lists and economic indexes have been developed and used largely by practicing managers and their staffs. Portfolio models have largely been developed by operations research specialists. Portfolio models deal with the problem of determining optimum funding allocations.

18. MILITARY:

A target defense problem is a very crucial problem in military especially at the time of war. H is desired to select numbers of areas and point interceptors that minimize the cost of defensive missiles under the maximum total expected damage produced by an unknown number of attacking missiles. This problem of minimizing the cost can be formulated as a mathematical programming problem. Also in planning future antiballistic missile deployments it is desirable to model the problem of determining optimal deffensive deployments under some restrictions.

Caterers problem can also be viewed as a military problem where the napkins are the bombart aircrafts and laundering is the repairing of these aircrafts under normal situation and at emergency period.

In the subsequent chapters some of the above problems and their solutions are discussed in detail.

CHAPTER - II

CONSTRUCTION OF BLOCK DESIGNS USING MATHEMATICAL PROGRAMMING

2.1 INTRODUCTION : The problems arising in Design and analysis of experiments may be broadly classified into following four categories :-

i) Construction of design with desired properties.

ii) Choosing optimal design with respect to a given criterion

iii) Choosing a suitable model and

iv) Analysing the experimental data.

The choice of optimal design and the model influence the conclusions drawn from the experiment. In this chapter the problem of construction of optimal design with respect to a given optimality criterion is viewed as a problem of Mathematical programming problem.

2.2 PRELIMINARIES : Experiment means a trial or special observation made to confirm or disprove something doubtful, especially one under conditions determined by the experimenter and to discover some unknown principles or effects. A particular set of experimental conditions is called a treatment. The term experimental unit is used to denote the material to which a treatment is applied in a single trial. Replication means repetition of treatments to several experimental units. A block is a set (group) of experimental units into which a particular treatment is applied to one experimental unit only. If the number of experimental units in a block is equal to the number of treatments the block is said to be complete, otherwise it is said to be incomplete. The conclusion from the experiments are affected not only by the assignable causes (such as treatments, blocks etc), but also by extraneous causes (beyond our control) which is termed as experimental error.

The simplest design is that in which treatments are allocated to the experimental units by a random process, such a design is called a completely randomized design (CRD). In a CRD there is no restriction on the replication of the treatments to the experimental units. CRD is suitable if the experimental units are homogeneous. The response obtained from an experimental unit with treatment 'k' differs from the response with treatment 1, by a constant, $\tau_{k-\tau i}$, The objective of the experiment is to estimate such differences.

The mathematical model for CRD is

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij} \qquad \begin{array}{l} i = 1, \ ....t \ .... \ (2.2.1) \\ j = 1, \ ....n_i \end{array}$$

where μ is the true mean effect, τi is the true effect of the i^{th} treatment, and ϵ_{ij} is the experimental error arising when the treatment i is applied to experimental unit j. Here we assume that the response obtained on one unit is unaffected by the treatment applied to another unit. The model (2.2.1) is called a fixed effect one-way classification model if we assume that

τ_i are randomly selected from a population of treatments, then we have a random effect model, In such cases the usual assumption is that it follows normal distribution with mean zero and variance $\delta\tau^2$.

In matrix notation (2.2.1) can be written as,

$$Y = XQ + \epsilon \text{ ....}(2.2.2)$$

where Y is the vector of responses, and X is the matrix of zeros and ones, called the design matrix.

$$\Theta' = (\mu, \tau_1, \tau_2 \text{ .... } \tau_t)$$

and $\epsilon' = (\epsilon_{11}, \text{ .... } \epsilon_{tn})$

If the experimental units are not homogeneous, then CRD is not suitable. In such situations the experimental units are divided into various homogeneous groups called blocks with the restriction that the number of units in each block is same as the number of treatments, and treatments are allotted by a random process to each experimental unit independently in each block. Such a design is called randomized block design (R B D).

The mathematical model of the randomized block design is,

$$Y_{ijk} = \mu + \tau_{i+pj} + \epsilon_{ijk} \qquad \begin{array}{l} i = 1 \text{ ......} a \\ j = 1 \text{ ......} b \\ k = 1 \text{ ......} n_{ij}. \end{array} \qquad (2.2.3)$$

Where μ is the true mean effect, τ_i is the i^{th} treatment effect, B_j is the true effect of the j^{th} block and ϵ_{ijk} is the error.

The model given in (2.2.3) is known as two-way classification fixed effect model. If τ_i's are random then the model is known as random effect model.

In matrix notation the model can be expressed as,

$$\underline{Y} = \underline{X\Theta} + \underline{\epsilon} \quad \ldots\ldots\ldots\ldots\ldots (2.2.4)$$

where $\underline{\Theta}' = (\ \mu,\ \tau_1,\ \tau_2\ \ldots.\ \tau_a,\ \beta_1 \ldots.\ \beta_b)$

If we wish to investigate simultaneously the effect of several conditions on a given process, each treatment consists of all combinations that can be formed different factors. The possible conditions of a factor are called the levels of that factor, such experiments are called factorial experimental. The combination of two or more levels of more than one factor are the treatments. For example in an agricultural experiment with two factors (i) nitrogen fertilizer at two levels denoted by n_0 and n_1, and (ii) irrigation at two levels, I_o and I_1, we can form the following four combinations I_on_o, I_on_1, I_1n_0, I_1n_1. Such combinations form treatments in factorial experiments. The factorial experiments may be conducted in any of the basic designs.

2.3 CONSTRUCTION OF BALANCED INCOMPLETE BLOCK DESIGN

The precession of the estimate of a treatment effect depends on the number of replications of the treatment. That is larger is the number of replications, greater is the precision. Suppose in a design there are p blocks in each of which two treatments occur together, there the pair of treatments is said to be replicated p times. The precision of the estimate of the difference between two treatments depends on the number of replications of the two treatments.This consideration has been exploited to construct designs for varietal trials with larger numbers of treatments so as to reduce the block size.

There are situations in which it is not possible to have blocks of sizes equal to the number of treatments, i.e. in a block the number of units or plots is lesser than the number of treatments, then the block is said to be incomplete and a design with such blocks is called incomplete block design. Let v denote the number of treatments in an experiment and k denote the number of experimental units in each of the blocks (k is called the block size of the design). In an incomplete blocks design $K < V$. In order to ensure equal or nearly equal precision of the comparisons of different pairs of treatments, the treatments are so allocated to the different blocks that each pair of treatments has the same or nearly the same number of replications and each treatment has an equal number of replications say r.

Different patterns of values of the numbers of replications of different pairs of treatments in a design, have given rise to different types of incomplete block designs. When the number of replications of all pairs of treatments in a design is the same, then an important series of designs known as balanced incomplete block design (BIBD) is obtained. This series of designs ensures equal precessions of the estimates of all pairs of treatment effects. It was first devised by Yates (1936) for agricultural experiments. These designs have evidently some constructional problems. As the allotment of K of the v treatments in different blocks such that each pair of treatment is replicated a constant number of times is not straight-forward. The constructional problems were first solved by the joint efforts of Fisher, Yates and Bose (1939).

It was found that BIBD is not always suitable for varietal trials because these designs require large number of replications and further, suitable designs are not available for all numbers of treatments. To overcome such difficulties Yates (1936) evolved a series of incomplete block designs which he called lattice designs.

2.4 FORMULATION OF THE PROBLEM :

Let V = {1, 2, ...v} be the set of treatments and let $v\Sigma k$ be the set of all distinct subsets of size K based on V. Let $v\epsilon k$ denote the cardinality of $v\Sigma k$. Let b denote the total number of blocks in the design. A BIBD, d, with parameters v, b, r, k and λ denoted by BIBD (v, b, r, k, λ) is a collection of b elements of $v\Sigma k$ (not necessarily distinct), called blocks, with these properties,

i) each element of V occurs in exactly r blocks, and

ii) each pair of distinct elements of V appears in exactly λ blocks.

Thus a BIBD (v, b, r, k, λ) is a combinatorial arrangement of v treatments in b blocks, containing k experimental units in each. The v treatments occur in such a way that each treatment does not occurs more then once in any block, each treatment occurs on r experimental units and each pair of treatment occurs λ times.

It is necessary for v,b,r,k and λ to satisfy the following relations.

$$i)\ bk = vr \ldots\ldots\ldots\ldots\ldots\ldots (2.4.1)$$

$$ii)\ \lambda\ (v-1) = r(k-1) \ldots\ldots\ldots (2.4.2)$$

$$iii)\ b \geq v \ldots\ldots\ldots\ldots\ldots\ldots (2.4.3)$$

These conditions are not sufficient for the existence of a BIBD (v,b,r,k,λ). Therefore we have to find ways to construct such designs. There is no single general method for the construction of all BIBD. There are specific methods for specific series of designs. These are tabulated by Fisher and Yates (1948), Rao (1961) and Sprott (1962).

We have not explicitly restricted b blocks to be distinct. So some of the blocks corresponding to certain element of v Σ k may occur more than once. By repeating p times the b blocks of a BIBD with parameters v,b,r,k and λ another BIBD with parameters v, pb, k, pr, pλ is obtained. Such designs are called repeated designs. Such designs are of practical significance as they allow us to restrict certain treatment combination being excluded from the experiment, for various considerations.

A BIBD, d, is said to be uniform if the distinct elements of vΣk appearing in the blocks b are such that they are repeated the same number of times in b. If b<vck then it is said to be reduced BIBD.

In the following we will construct BIBD with repeated blocks as an integer programming problem given by Arthanari and Dodge (1981).

Let F be a frequency vector corresponding to a BIBD where $F'=(f_1, \ldots\ldots f_{vck})$ is such that fi is the frequency of the ith element of vΣk.
Now let,

$$\sum_{i=1}^{vck} f_i = b$$

and b^* be the number of nonzero entries in F let the elements of vΣk be numbered from 1 to vck.

Matrix P is called a pair inclusion matrix for a given v and k in case $P_{ij} = 1$ if the i^{th} element of vΣ2 is contained in the j^{th} element of vΣk, and $P_{ij}=0$, otherwise, we call the vector Pj′ $= (P_{1j}, P_{2j}, \ldots. P_{vc2})$ the pair inclusion vector associated with $j \in$ vΣk.

Thus $P = [P_1 \ldots\ldots P_{vck}]$. Let e denote the vector $(1\ldots\ldots1)'$ with vc_2 elements. We have the following result.

<u>LEMMA</u> : Any frequency vector F corresponding to a BIBD of BIBD (v, b, r, k, λ) satisfies,

$PF = \lambda e$ ....(2.4.4)

$F \geq 0$, integer vector ....(2.4.5)

and any integer F satisfying (2.4.4) and (2.4.5) is a frequency vector corresponding to a BIBD (v, b, r, k, λ).

(The proof may be seen in Arthanari & Dodge (1981) page 312-313).

Thus the problem of constructing BIBD based on v, k and λ is formulated as the problem of finding feasible solution to an integer programming problem. In fact, if we consider $PF-\lambda e = 0$ and find integer solution to this homogeneous system of equation with (F, λ) integers we have a BIBD.

Whenever we have a rational solution to the system we have an integer solution as well. We find such a solution by taking the least common factor (lcm) of the denominators of the f_j's and multiplying f_j's by the lcm, for this modified F, λ will be an integer.

2.5 <u>AN EXAMPLE</u> : Let v=5 and k=3. The pair inclusion matrix P for these parameters are,

Para	1,2,3	1,2,4	1,2,5	1,3,4	1,3,5	1,4,5	2,3,4	2,3,5	2,4,5	3,4,5
1 2	1	1	1	0	0	0	0	0	0	0
1 3	1	0	0	1	1	0	0	0	0	0
1 4	0	1	0	1	0	1	0	0	0	0
1 5	0	0	1	0	1	1	0	0	0	0
2 3	1	0	0	0	0	0	1	1	0	0
2 4	0	1	0	0	0	0	1	0	1	0
2 5	0	0	1	0	0	0	0	1	1	0
3 4	0	0	0	1	0	0	1	0	0	1
3 5	0	0	0	0	1	0	0	1	0	1
4 5	0	0	0	0	0	1	0	0	1	1

The homogeneous equations given by

PF = λe. are where $F' = (f_1 \;\ldots\ldots\; f_{10})$
$e' = (1,\ 1\ \ldots.\ 1)$
$\lambda = 3$

are
$$f_1 + f_2 + f_3 = 3$$
$$f_1 + f_4 + f_5 = 3$$
$$f_2 + f_4 + f_6 = 3$$
$$f_3 + f_5 + f_6 = 3$$
$$f_1 + f_7 + f_8 = 3$$
$$f_2 + f_7 + f_9 = 3$$
$$f_3 + f_8 + f_9 = 3$$
$$f_4 + f_7 + f_{10} = 3$$
$$f_5 + f_8 + f_{10} = 3$$
$$f_6 + f_9 + f_{10} = 3$$

The problem given by (2.4.4) and (2.4.5) is equivalent to an integer programming problem with 0-1 matrices and constant right-hand side vector.

Let A be an mxn matrix, b an mx1 vector and C be a 1 x n vector. Then the integer programming problem is defined as,

$$\begin{aligned} &\text{Minimize } C'X \\ &\text{Subject to} \quad AX = b \\ &\qquad\qquad\quad X \geq 0 \\ &\qquad\qquad\quad X \text{ are 0 or 1} \end{aligned}$$

In our problem we have the following correspondence, A=P, m=v(v-1)/2, n=vck, X is denoted by F, λe a constant vector is equal to b.

For the above integer-programming problem we have the following results :

<u>RESULT 1</u> : A subset M of the index set of columns from matrix A is called a represent of A if and only if

$$A_M e \geq e$$

where A_M is the submatrix obtained from A corresponding to the elements in M and M is a feasible solution.

A represent M of A is minimal of no M', M' = M - {j} for any $j \epsilon M$, is a represent of A.

Let M be a minimal represent of A. There exist X_M such that $A_M X_M = \lambda e$ for some λ positive integer iff $A_M e = e$.

With the correspondence of integer programming and construction of BIBD we can observe that,

i) the existence of $X_M > e$ such that $A_M X_M = \lambda e$ is equivalent to saying that M is a support of a BIBD, and

ii) $A_M e = e$ is equivalent to saying that M is itself a BIBD with $\lambda = 1$.

The following results connect minimal represents and support of BIBD.

<u>RESULT 2</u> : If M is a minimal represent of P then M is the support of a BIBD iff M is itself a BIBD for all λ , positive integer.

<u>RESULT 3</u> : BIBD with repeated blocks for all λ exist iff there exist a minimal represent of P which is the support of BIBD.

2.6 <u>ANOTHER APPROACH OF CONSTRUCTING BIBD</u> :

Whitaker, Triggs and John (1990) constructed optimal or nearly optimal block designs using an optimality creterion termed as 'A-optimality' criterion. They formulated the problem as a nonlinear 0-1 programme.

A block design is said to be of size (v,k,r) if v treatments are arranged in b blocks each of size k with r replications of each treatment. In the above situation $vr=bk$. The following assumption are made,

i) $k < v$

ii) no treatment occur more than once in a block

iii) all the treatments are of equal interest

iv) the design is connected and every treatment contrast is estimable.

Let n_{ij}, $i=1, \ldots\ldots v$, $j=1, \ldots\ldots b$, denote the number of times the i^{th} treatment occurs in the j^{th} block. The assumptions (i) and (ii) leads us to the following conclusions,

$$\left.\begin{aligned} & n_{ij}=0 \text{ or } n_{ij}=1 \\ & \sum_{j=1}^{b} n_{ij}=r,\ i=1..v. \\ & \sum_{i=1}^{v} n_{ij}=k\ \ j=1....b \end{aligned}\right] \quad .... (2.6.1)$$

n_{ij}'s can be arranged as a v x b matrix.

$N = \{N_{ij}\}$

If λ_{ij}, $(i,j=1,2...v)$ is defined as

$\lambda_{ij} = r$ for $i = j$

and λ_{ij} = frequency of ith and j^{th} treatments appearing together in a block for $i \neq j$, then the λ_{ij}'s can also be arranged as a v x v matrix $NN' = \{\lambda_{ij}\}$. If λ_{ij}, $i \neq j$ are equal to their average $\lambda = r(k-1)/(v-1)$, then the design is a balanced incomplete design.

A necessary, though not sufficient, condition for the existence of a BIBD is that λ be an integer.

Optimality criteria for choosing between different designs of the same size can be based on the nonzero eigen values e_1, e_2 ..... e_{v-1} of the matrix.

$C = I-NN'/rk$ .... (2.6.2)

These eigen values are the canonical efficiency factors. A design is said to be A-optimal if the harmonic mean of the cannonial efficiency factors is greater than or equal to that of any other design of the same size.

An A - optimal design maximizes

$$E = (v-1)/\Sigma e_i^{-1} \quad .... \quad (2.6.3)$$

Statistical justification for using this criterion is that it minimizes the average standard error of treatment differences $(2\delta^2/rE)^{1/2}$

Eigen value computations are time consuming, thus, a simpler criteria have been proposed which initially sift out the less efficient designs and then use E, if necessary to choose among the remaining. Shah (1960) and Eccleston and Hedayat (1974) use (M,S)-optimality creterion, which for binary designs is equivalent to minimizing trace $(NN')^2$. Hall and Jarrett (1983) have chosen designs from the (M,S) - optimal class by minimizing trace $(NN')^3$. A general procedure is obtained by sequentially minimizing the traces of higher powers of (NN').

The problem of finding designs using A-optimality creterion can be expressed as the following nonlinear 0-1 programming problem.

Minimize $f_s = g(n_{ij})$ .... (2.6.3)

Subject to

$$\left.\begin{array}{l} \sum_{j=1}^{b} n_{ij} = r \quad i=1\ldots.v \\ \\ \sum_{i=1}^{v} n_{ij} = k \quad j=1\ldots b \\ \\ n_{ij}=0 \quad \text{or} \quad 1 \quad \forall\, i,j \end{array}\right] \quad \ldots.(2.6.4)$$

where $f_s = \Sigma(rk)^{-1}\, T_i$

T_i = trace of $(NN')^i$

$$= \sum\sum\ldots\sum \lambda_{m_1m_2}\lambda_{m_2m_3}\ldots\ldots\lambda_{m_xm_1}$$

$$\text{and}\ \lambda_{ij} = \sum_{p} n_{ip}n_{jp}$$

The constraint n_{ij} = 0 or 1 may be replaced by the constraint.

$$= \sum_i \sum_j n_{ij}(1-n_{ij}) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2.6.5)$$

and $n_{ij} \le 1$

The problem can now be stated as,

$$\text{Minimize } \Sigma(rk)^{-1}Ti \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.6.6)$$

$$\left.\begin{array}{ll} \text{Subject to } {}_j\Sigma\, n_{ij} = r & i = 1 \ldots v \\ {}_i\Sigma\, n_{ij} = k & j = 1, .b \\ \sum\Sigma n_{ij}(1-n_{ij}) = 0 \text{ for } \forall i,j & \\ \text{and } n_{ij} \leq 1 \text{ for } \forall\, i,j & \\ n_{ij} \geq 0 \quad \text{for } \forall\, i,j & \end{array}\right] \quad \ldots\ldots (2.6.7)$$

The above problem is a large problem in vb variables, v+b+1 constraints and vb upper bounds. Fortunately the set of feasible solution is sparse enough allowing very large designs to be generated. The above problem may be solved by using any suitable nonlinear programming computer package. Whitaker, Triggs & John (1990) solved the problem (2.6.6) - (2.6.7) using nonlinear programming package MINOS.

CHAPTER III

NETWORK FLOW IN MANPOWER SCHEDULING

3.1 INTRODUCTION :

The various operations (activities or jobs or tasks) necessary to complete a project and the order in which the various operations are to be performed may be shown in a project graph, called a network. In a network, the various operations or jobs are shown by arrows leading from one circle in the graph to another. These circles are called "nodes". The aero indicates the direction of progress of the project. Many aspects in various walk of life are frequently concerned with the analysis of problems which are defined by a set of simulatenous linear equations and are sparse; that is they have a large number of zero coefficients. In general these equations can be mathematically modelled as a network and solved by the network approach. The problem to which these modern technique can be applied are related to physical network such as electricity supply, gas distribution, water distribution, communication system, mechanical and civil engineering structures, traffic flow, man-power scheduling etc.

3.2 PRELIMINARIES :

Consider a network consisting of a finite number of nodes $N=\{1,2, \ldots\ldots\ldots\ldots m\}$ and a set of directed arcs (lines) $S=\{(i,j), (k,l) \ldots\ldots (s,t)\}$ joining pairs of nodes. Arc (i,j) is said to be incident with nodes i, and j and is directed from node i to node j. We assume that the network has m nodes and n arcs. With each node i in the network we associate a number bi, that is the available supply of an item (if bi > 0) or the required demand for the item (if bi < 0). Nodes with bi > 0 are called sources, and nodes with bi < 0 are called sinks. If $b_i = 0$, then none of the item is available at node i and none is required. In this case node i is called an intermediate (or transhipment) node. Let X_{ij} be the amount of flow on the arc (i, j) and Cij be the shipping cost along the arc. Assume that the total supply is equal to the total demand within the network, that is

$$\sum_{i=1}^{m} b_i = 0$$

The minimal cost network flow problem may be stated as follows. Ship the available supply through the network to satisfy demand at minimal cost. This problem may be formulated as the following mathematical programming problem:

Minimize

$$\sum_{i=1}^{\dots} \sum_{j=1}^{\dots} C_{ij} x_{ij}$$

Subject to,

$$\sum_{j=1}^{m} x_{ij} - \sum_{k=1}^{m} x_{kl} = b_i \quad i=1\dots m$$

$$x_{ij} \geq 0 \; i, \quad j=1,2\dots\dots\dots\dots\dots\dots m$$

The m constraints are called the flow conservation or Kirchhoff equations. They indicate that the flow may be neither created nor destroyed in the network.

In the conservation equations,

$$\sum_{j=1}^{m} x_{ij}$$

represents the total flow out of node i while

$$\sum_{k=1}^{m} x_{kl}$$

indicates the flow into node i. These equations require that the net flow out of node i,

$$\sum_{j=1}^{m} x_{ij} \sum_{k=1}^{m} x_{kl}$$

should be equal to bi. If bi < 0, then there should be more flow into i than out of i.

The minimal cost flow problem is a linear programming problem and can be solved by any one of several available techniques. One way is to apply the ordinary simplex algorithm to the problem.

<u>PATH</u> : A path from node i_0 to i_p is a sequence of arcs P = $\{(i_0, i_1), (i_1, i_2), \ldots\ldots (i_{p-1}, i_p)\}$ in which the initial node of each arc is the same as the terminal node of the preceding arc in the sequence. Thus each arc in the path is directed toward i_p and away from i_0.

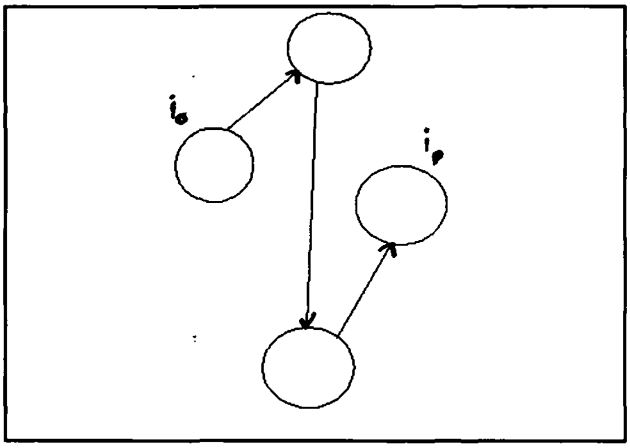

FIG: (3.2.1) - A Path

CHAIN : A chain has a similar structure as a path except that not all arcs are necessarily directed towards node i_p.

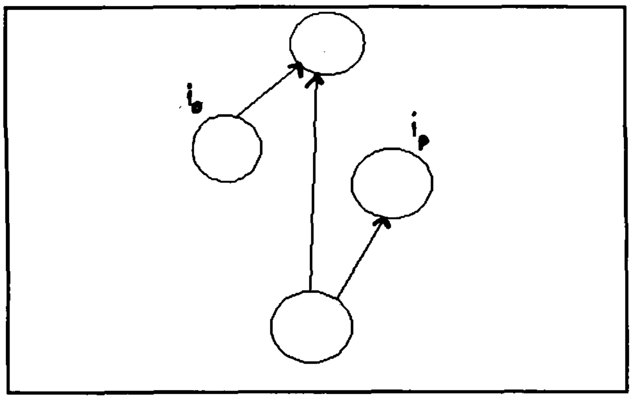

FIG : (3.2.2) A Chain

CIRCUIT : A circuit is a path in which $i_0 = i_p$. Thus a circuit is a closed path.

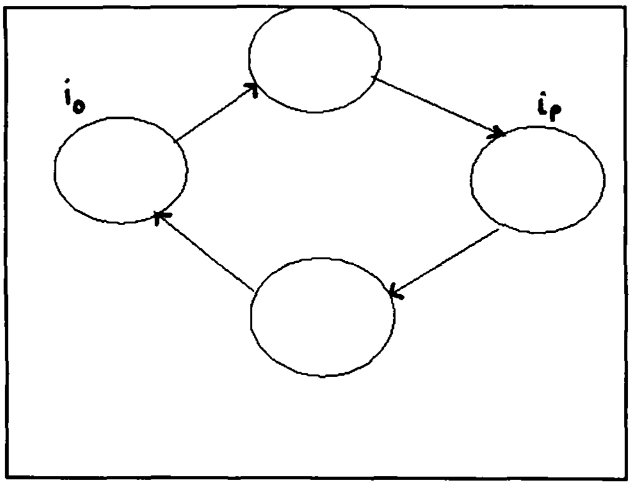

Fig.:(3.2.3) A Circuit

CYCLE : A cycle is a clossed chain.

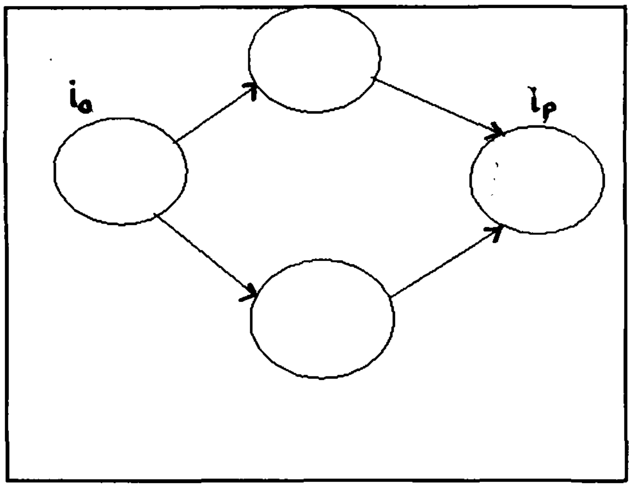

Fig : (3.2.4) A Cycle

Now a days the network models are the most powerful management tool available for solving wide variety of optimization problems efficiently and at a low cost. Numerous networks have been developed with varying degrees of applicability. The Out-of-Kelter method of Ford and Fulkerson (1962) is used to solve the maximal value of flow problem. A Branch-and-Bound algorithm is constructed by Florian and Robillard (1971) for computing minimum cost flows in capacitated networks with concave cost functions Wright (1975) employes a network algorithm to facilitate the reallocation of housing to tenants of a housing authority. A class of transportation scheduling problems were addressed by Gavish & Shlifer (1979). Peterson (1980) formulate multicommodity flow problem replacing conservation of flow constraints by flows across cuts.

Consider a man-power scheduling problem in a fire department that may be treated as a network flow problem. The objective of the problem is to minimize the flow of manpower through the network while the system constraints are satisfied. Khan M. R. (1981) gave the minimum flow algorithm which has computational advantages over the Simplex method.

3.3 <u>THE PROBLEM</u> :

Suppose the fire department in a small city consists of two zones denoted by Z_1 and Z_2. The department operates on a three-shift basis and assigns in each shift a number of fireman to each zones. The necessary data is provided in Table 3.3.1. Further constraints require that the zones must have atleast 14 and 23 fireman per 24 hours respectively. The problem is to determine the number of men committed to the field with both zones adequately covered in each shift.

Shift	Minimal no.of fireman required	Minimal Maximal no. of fireman in zone	
		Z_1	Z_2
1	8	3 - 5	5 - 7
2	15	4 - 6	10 - 12
3	18	7 - 9	9 - 12

Table 3.3.1

The above scheduling problem is represented by the network in Figure (3.3.1). The nodes 2 and 3 denote the zones, and nodes 4, 5 and 6 denote the shifts. The manpower resource, as a flow, has to be moved from node 1 (source) to node 7 (sink).

The indicated capacities within parenthesis along the arcs are the minimum and the maximum flow limits on the arcs. Whenever the upper bound on the flow is not satisfied, this limit is assumed to be α. The objective is to find the minimal flow through the network subject to the specified constraints.

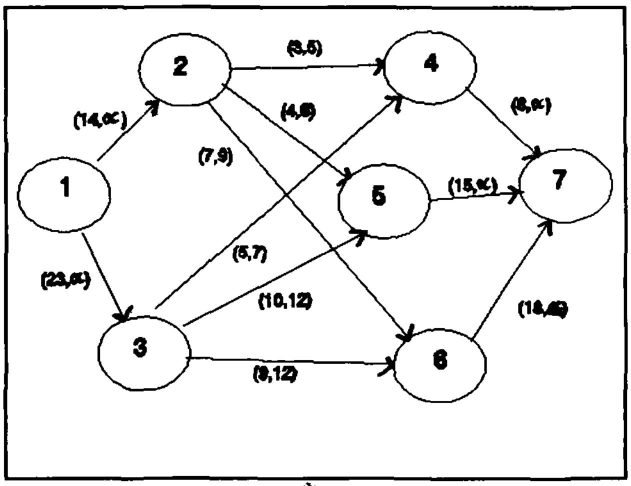

Fig. 3.3.1

Let X_{ij} denote the flow through the arc joining nodes i(=1,2,3,4,5,6) and j(=2,3,4,5,6,7).

3.4 THE ALGORITHM :

An algorithm for solving the above manpower scheduling problem, studied by Khan (1981) is presented below. Let

r_{ij} = minimum flow required between nodes i & j.

k_{ij} = maximum flow allowable between nodes i and j, obviously $K_{ij} \geq k_{ij}$

x_{ij} = actual flow between nodes i and j.

f_{ij} = current limit on minimum flow between nodes i and j.

f_{ij} = current limit on maximum flow between nodes i and j.

r = increment in flow along the selected path.

Thus the solution of this problem as a minimal flow through the network must satisfy the following constraints :

$k_{ij} \leq X_{ij} \leq K_{ij}$ for all (i, j) .... (3.4.1)

The various steps of the algorithm are,

<u>Step 1</u> : Set r = 0 and $X_{ij} = 0$

<u>Step 2</u> : Set the current minimum/maximum limits.

$f_{ij} = k_{ij} - r$ for all (i, j)

$F_{ij} = k_{ij} - r$ for all (i, j)

<u>Step 3</u> : Find any arc in the network whose lower limit is still unsatisfied (i.e. $f_{ij} > 0$). If all the arcs have been satisfied go to step 8; otherwise go to step 4.

<u>Step 4</u> : Identify all paths in the network associated with arc in step 3, and select the path that has :

a) $F_{ij} > 0$ for all arcs along the path (break a tie arbitrarily, if any) and
b) the most number of arcs for which $f_{ij} > 0$ (break a tie arbitrarily, if any).

<u>Step 5</u> : Find the minimum required flow r along the selected

path in step 4, but not exceeding any upper limit, that is;

$$r = \text{minimum} \quad \begin{cases} \{f_{ij} > 0\}, \ (i, j) \text{ along the path} \\ \{F_{ij} > 0\}, \ (i, j) \text{ along the path} \end{cases}$$

Step 6 : Increase the flow along the path in step 4, by the amount r in step 5, and compute the upper and lower limits f_{ij} and F_{ij} as :

$$f_{ij} = f_{ij} - r, \ (i, j) \text{ along the path}$$

$$F_{ij} = F_{ij} - r, \ (i, j) \text{ along the path}$$

Step 7 : Set the actual flow to each arc as :

$$X_{ij} = k_{ij} - f_{ij}$$

Step 8 : Check the constraint (3.4.1) for the feasibility of the current solution.

If (3.4.1) is satisfied for all the arcs in the network, an optimal solution has been found, otherwise go to step 3.

In the procedure mentioned above we have to find an arc in the network which requires increase in the flow, then find the minimum allowable flow along the path associated with this arc and has the most number of unsatisfied arcs, and increase the flow along this path by that amount. This will specify new limits on the arcs. We will repeat this process for every arc whose lower limit is not yet satisfied till flow along each arc meets the constraints (3.4.1).

3.5 THE SOLUTION :

The solution to the sample problem given in (3.3) by the application of the algorithm described in section (3.4) is given below :

Let us choose any arc in the network whose lower limit is not satisfied ($f_{ij} > 0$). For example in our problem we have two arcs (1, 2) or (1, 3) which requires an increase in the flow since the lower limits are not satisfied {i.e. $f_{12} = 14 > 0$, $f_{13} = 23 > 0$}. Lets us select the arc (1, 2). Three paths (1-2-4-7), (1-2-5-7) and (1-2-6-7) are associated with this arc. None of the arcs on these paths has its upper bound reached {i.e. $F_{12} = \alpha > 0$, $F_{24} = 5 > 0$, $F_{57} = \alpha > 0$ etc }. So that a flow along any of these paths is permissible. Further each path has an equal number of arcs whose lower bounds are still unsatisfied. Thus we select any of these paths arbitrarily such as path (1-2-4-7).

We find the minimum allowable flow along this path by calculating r as :

$$
\begin{aligned}
r &= \text{minimum } \{f_{12},\ f_{24},\ f_{47},\ F_{12},\ F_{24},\ F_{47}\} \\
&= \text{minimum } \{14,\ 3,\ 8,\ \alpha,\ 5,\ \alpha\} \\
&= 3
\end{aligned}
$$

The updated bounds on each arc along the path is obtained by subtracting r = 3 units from the previous bounds.

Thus the new bounds for the arc (1,2), (2,4) and (4,7) are (11, α), (0, 2) and (5, α) respectively.

For the arc (2,4) the lower limit indicates that this arc has already been satisfied but not the others.

Next, let us select the path (1-2-5-7). The minimum allowable flow along this path is,

$$r = \text{minimum } \{f_{12}, f_{25}, f_{57}, F_{12}, F_{25}, F_{57}\}$$
$$= \text{minimum } \{11, 4, 15, \alpha, 6, \alpha\}$$
$$= 4$$

The new upper and lower bounds on this path are (7,α), (0,2) and (11,α) for the arcs (1,2), (2,5) and (5,7).

Selecting the path (1-2-6-7), since it has the maximum number of unsatisfied arcs, the minimum allowable flow is,

$$r = \text{minimum } \{f_{12}, f_{26}, f_{67}, F_{12}, F_{26}, F_{67}\}$$
$$= \text{minimum } \{7, 7, 18, \alpha, 9, \alpha\}$$
$$= 7$$

The new limits for the arc (1,2) is (0,α) for (2,6) is (0,2) and that for (6,7) is (11,α).

For the arcs (1,2), (2,4), (2,5) and (2,6) the lower limit is zero which indicates that these arcs are satisfied.

Now, let us choose the arc (1,3) and the path (1-3-4-7) and increase the flow by,

$$r = \text{minimum } \{f_{13},\ f_{34},\ f_{47},\ F_{13},\ F_{34},\ F_{47}\}$$
$$= \text{minimum } \{23,\ 5,\ 8,\ \alpha,\ 7,\ \alpha\}$$
$$= 5$$

The new limits are $(18,\alpha)$, $(0,2)$ and $(0,\alpha)$ for the arc (1,3), (3,4) and (4,7) respectively.

Along the path (1-3-5-7) which is also associated with the arc (1,3), increase the flow by r=10 units. The new limits are $(8,\alpha)$, $(0,2)$ and $(1,\alpha)$ for the arcs (1,3), (3,5) & (5,7) respectively.

Let us take arc (1,3) again and the path (1-3-6-7) and increase the path by r=8 units. The new limits are $(0,\alpha)$, $(1,4)$ and $(3,\alpha)$ for the arcs (1,3), (3,6) and (6,7) respectively.

The arcs (1,3), (3,4), (4,7) and (3,5) are all satisfied. Since the arc (3,6) is still unsatisfied, we will consider this arc to increase the flow. The path with this arc, let us select (1-3-6-7). The minimum allowable flow in this path is r=1. The new limits are $(-1,\alpha)$, $(0,3)$ and $(2,\alpha)$. The limit $(-1,\alpha)$ on the arc (1,3) implies that the current flow through this arc is 1 unit in addition to its minimum requirement, namely 23-(-1)=24 units.

Next let us select the arc (6,7) and increase the flow by r=2 along the path (1-2-6-7). The new limits are $(-2,\alpha)$, $(-2,0)$

and $(0,\alpha)$ for the arcs (1,2),(2,6) and (6,7) respectively.

Now the only unsatisfied arc is (5,7). Let us select this arc and the path (1-2-5-7) to increase the flow. The minimum increase in flow in this path is r=1. The new limits are, $(-3,\alpha)$, (-1,5) and $(0,\alpha)$.

The limit (-2,0) on arc (2,6) means that the current flow across this arc is 2 units over and above its minimum requirement, namely, (7-(-2)=9 units. Thus the upper limit 0 would not allow any further increase in the flow along this route in the future iterations.

All the arcs in the network are now satisfied i.e. the current solution has $f_{ij} \leq 0$ for all (i,j). The current flow across each arc is as follows :

$X_{12} = 14 - (-3) = 17$

$X_{13} = 23 - (-1) = 24$

$X_{24} = 3$

$X_{25} = 4 - (-1) = 5$

$X_{26} = 7 - (-2) = 9$

$X_{34} = 5$

$X_{35} = 10$

$X_{36} = 9$

$X_{47} = 8$

$X_{57} = 15$ and

$X_{67} = 18$

These flows satisfy the constraint (3.4.1) i.e.

$k_{ij} \leq X_{ij} \leq K_{ij}$ for all (i,j)

Thus the above solution is an optimal solution. The above results are tabulated as :

Shift	Zone		Total
	Z_1	Z_2	
1	3	5	8
2	5	10	15
3	9	9	18
TOTAL	17	24	41

Table 3.5.1

The minimal flow through the network system is,

$X_{12} + X_{13} = 17 + 24 = 41$ units (fire man)

3.6 OPTIMALITY OF THE SOLUTION :

The network flow algorithm described in (3.4) converges to an optimal solution to the minimal network flow problem, provided such a solution exists.

Suppose that the process is at the K-th iteration of the algorithm. At this iteration, let r be the minimum required flow for the selected path according to step 4 of the algorithm. Let $f^k_r(i,j)$ and $F^k_r(i,j)$ be the limits of the increased flow between nodes i and j, as defined in the following manner :

$$f^k_r(i,j) = f^k\ (i,j) - r,\ (i,j) \text{ along the path}$$

$$F^k_r(i,j) = F^k\ (i,j) - r,\ (i,j) \text{ along the path}$$

when $f^k\ (i,j)$ and $F^k\ (i,j)$ denote the current limits on minimum and maximum flow between nodes i and j at the k^{th} iteration of the process. Then we have the following result:

<u>LEMMA</u> : Let $f^k\ (i,j)$, $F^k\ (i,j)$, $f^k_r(i,j)$ and $F^k_r(i,j)$ be as defined above. Then, at the k^{th} iteration of the algorithm, we have,

$$f^k_r(i,j) > 0, \text{ if and only if } f^k\ (i,j) > r \ldots\ldots(3.6.1)$$

$$f^k_r(i,j) = 0, \text{ iff } f^k\ (i,j) > r \ldots\ldots\ldots\ldots\ldots(3.6.2)$$

$$f^k_r(i,j) < 0, \text{ iff } f^k\ (i,j) > r \ldots\ldots\ldots\ldots\ldots(3.6.3)$$

Similarly for $F^k_r(i,j)$.

<u>LEMMA 2</u> : Let $f^k_r(i,j)$ be the lower limit of the increased flow by r_0 between nodes i and j. Then $f^k_r(i,j)$ is strictly monotonic decreasing in r. That is,

$$f^k_{r''}(i,j) < f^k r'(i,j), \text{ if } r' \leq r''$$

$$r',\ r'' > 0$$

<u>Proof</u> : Let r" minimizes the required flow in step 5 of the algorithm, then the lower limit on the flow is,

$$f^k_{r''}(i,j) = f^k\ (i,j) - r'' < f^k\ (i,j) - r' = f^k_{r'}(i,j)$$

which implies that $f^k_{r''}(i,j) < f^k_{r'}(i,j)$

<u>Corollary</u> : Let $F^k_r\ (i,j)$ denote the upper limit of the

increased flow by r between nodes i and j. Then lemma 2 above holds by replacing $f_r^k(i,j)$ with $F_r^k(i,j)$.

Lemma 3 : If the lower limit of the increased flow between nodes i and j is non-positive, i.e, $f^k{}_r(i,j) \leq 0$, then there is a feasible flow between these nodes.

Proof : The actual flow in the algorithm is defined by :

$$X_{ij} = k_{ij} - f_{ij}$$

which, in view of $f^k{}_r(i,j)$, gives

$$\begin{aligned} X_{ij} &= k_{ij} - f^k{}_r(i,j) \\ &\geq k_{ij} \end{aligned}$$

since $f^k{}_r(i,j) \leq 0$ implies that $X_{ij} \geq 0$

Corollary 2 : Let $F^k{}_r(i,j) \leq 0$, i.e. the upper limit of the increased flow between nodes i and j is non-positive. Then Lemma 3. above holds by replacing $f^k{}_r(i,j)$ with $F^k{}_r(i,j)$.

Therefore the algorithm starts with an infeasible solution, and changes the basis at each iteration for possible improvement. Each time a minimal increment in the flow along a given path is allowed, which reduces the degree of infeasibility of the previous solution. Thus the algorithm influences the current solution to approach feasibility in a gradual manner. The process terminates as soon as the capacity limits on the arcs are satisfied. Thus, the feasible solution found at the terminal interaction of the process is also optimal.

3.7 LINEAR PROGRAMMING FORMULATION :

The problem (3.3) can be formulated as a linear programming problem and solved by Simplex method, where at each iteration the improved solution moves from one corner of the solution space to the another, and the process terminates when the lowest improved solution is identified. Linear programming formulation of the network problem represented by Fig.(3.3.1) is,

$$\text{Minimize } Z = X_{24} + X_{25} + X_{26} + X_{34} + X_{35} + X_{36}$$

Subject to,

$$
\begin{array}{llllllll}
X_{24} & + X_{25} & + X_{26} & & & & \geq & 14 \\
 & & & X_{34} & + X_{35} & + X_{36} & \geq & 23 \\
X_{24} & & & + X_{34} & & & \geq & 8 \\
 & X_{25} & & & + X_{35} & & \geq & 15 \\
 & & +X_{26} & & & +X_{36} & \geq & 18 \\
X_{24} & & & & & & \leq & 5 \\
 & X_{25} & & & & & \leq & 6 \\
 & & X_{26} & & & & \leq & 9 \\
 & & & X_{34} & & & \leq & 7 \\
 & & & & X_{35} & & \leq & 12 \\
 & & & & & X_{36} & \leq & 12
\end{array}
$$

$X_{ij} \geq 0$ for all (i,j)

The above model can be solved by using the Simplex algorithm artificial basis technique of linear programming. In the above problem we have 11 inequality constraints. Introducing 5-surplus and 6-slack variables the inequalities may be changed into equations increasing the number of variables to 17. The optimal solution is found in the ninth iteration of the algorithm. The solution thus obtained is,

$$X_{24} = 3$$

$$X_{25} = 5$$

$$X_{26} = 9$$

$$X_{34} = 5$$

$$X_{35} = 10$$

$$X_{36} = 9$$

The value of the objective function is,

$$3+5+9+5+10+9 = 41$$

which is same as obtained in section (3.5).

The algorithm proposed by Khan (1981) is computationally more efficient and less time consuming. The algorithm has considerable computational advantage over the other procedures as it utilizes the network formulation which is generally considered more efficient. The problem handles a modest sized problem very conveniently. For a problem with a large number of arcs, however the scanning of arcs and paths may become a time-consuming effort. But this difficulty can easily overcome by programming the algorithm on computer.

Khan and Lewis (1987) applied the algorithm for staffing schedule of hospital nurses, which has become critical in recent years for the efficient utilization of nursing staff.

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CHAPTER IV

OPTIMAL INTEGRATION OF SURVEY

4.1 INTRODUCTION : In sample surveys with varying probabilities of selection of various population units may be taken as proportional to some measure of their size, this is known as probability proportional to size (pps) sampling. If more than one sample surveys are to be conducted on the same population to estimate different characteristics of the population one measure of size is not usually suitable for all characteristics. Thus one has to use different measures for different characteristics and consequently has to draw separate samples for each survey, which will result in the increase of the total cost of surveys. Let some units of the population appear in all the samples drawn for different surveys, without violating the probability restrictions on selection of samples. This will result in the reduction of the total cost of the survey. Thus one man be interested in maximizing the expected number of common units on the selected samples for the different surveys, this is termed as "integration of sample surveys".

4.2 SELECTION SCHEMES : In the following, three selection schemes for integration of sample surveys have been discussed. For the sake of simplicity we assume that only two surveys "Survey I" and "Survey II" are to be conducted on the population having

N units u_1, u_2 .... u_n. Let sample - I and Sample - II, both of size n, be selected for Survey-I and Survey-II respectively.

Further let $p_i^{(1)}$ = probability of inclusion of u_i in sample-I

$p_i^{(2)}$ = probability of inclusion of u_i in sample-II

The inclusion of u_i in both the samples is termed as an overlap. Obviously, at most, there can be n overlaps. The problem is to maximize the expected number of overlaps such that u_i receive $p_i^{(1)}$ and $p_i^{(2)}$ as probabilities of selection in survey-I and survey-II respectively.

4.2.1 KEYFITZ'S SELECTION SCHEME :

Keyfitz (1951) suggested that the two surveys can be integrated by making the sample selection for the second survey dependent on that for the first survey. The different steps of the selection scheme are :

i) Select a sample of n distinct units for survey-I, with the prescribed probabilities $p_i^{(1)}$.

ii) Let u_i is selected in the sample-I, if $p_i^{(2)} \geq p_i^{(1)}$ retain u_i for the sample - II.

iii) If $p_i^{(2)} < p_i^{(1)}$ select u_i in sample-II with probability $p_i^{(2)}/p_i^{(1)}$.

iv) If $p_i^{(2)} < p_i^{(1)}$ and u_i is not selected in step(iii), select a unit u_j with probability proportional to $p_j^{(2)} - p_j^{(1)}$

among those for which $p_j^{(2)} > p_j^{(1)}$.

The following lemma shows that under the above scheme the units are selected in sample-I and sample-II with their appropriate probabilities.

<u>LEMMA</u> : Unit u_i is selected with the prescribed probability $p_i^{(1)}$ and $p_i^{(2)}$ for survey-I and survey-II respectively in Keyfitz's selection scheme.

<u>PROOF</u> : In Keyfitz's scheme the sample for survey-I is chosen so that unit u_i is selected in sample-I with probability $p_i^{(1)}$. So the probability restriction for the survey-I is satisfied.

The fact that the required probabilities $p_i^{(2)}$ for the units u_i, $i = 1, 2, ...n$ are achieved by this sampling scheme in respect of the survey-II can be proved by noting that when $p_i^{(2)} < p_i^{(1)}$, the unit u_i is selected in the sample-II if it is selected in the sample-I and it is retained with probability $p_i^{(2)}/p_i^{(1)}$, that is,

$$P(u_i) = p_i^{(1)}(p_i^{(2)}/p_i^{(1)}) = p_i^{(2)}$$

and that if $p_i^{(2)} > p_i^{(1)}$, u_i is selected in the sample-II if it is selected in the sample-I or if some unit u_j, $(j \neq i)$, with $p_j^{(2)} < p_j^{(1)}$ is selected in the sample-I and it gets rejected with probability

$$p_i^{(2)} - p_i^{(1)} / \sum_{p_j^{(2)} > p_j^{(1)}} (p_j^{(2)} - p_j^{(1)})$$

that is in this case.

$$P\ (u_i)\ = p_i^{(1)} + \sum_{p_j^{(2)} < p_j^{(1)}} p_j^{(1)}\ (1-p_j^{(2)}/p_j^{(1)})\ \frac{p_i^{(2)} - p_i^{(1)}}{\sum_{p_j^{(2)} > p_j^{(1)}} (p_j^{(2)} - p_j^{(1)})}$$

But

$$\sum_{p_j^{(2)} < p_j^{(1)}} (p_j^{(2)} - p_j^{(1)}) = \sum_{p_j^{(2)} > p_j^{(1)}} (p^{j(2)} - p_j^{(1)})$$

therefore $P(u_i) = p_i^{(1)} + p_i^{(2)} - p_i^{(1)} = p_i^{(2)}$ as desired.

Further, it can be seen that the probability of getting an overlap for the two surveys is

$\Sigma \min\ (p_i^{(1)}, p_i^{(2)})$ at each draw.

4.2.2 LAHIRI'S SELECTION PROCEDURE :

Lahiri (1954) suggested arrangement of the units in the sampling frame is a serpentine order so that any two geographical contiguous units occur next to each other in the sampling frame and then selection of units for the two surveys are done with the same set of n random numbers chosen from 0 to 1, but with independent cumulative totals of the probabilities $p_i^{(1)}$ and $p_i^{(2)}$, For instance, if r is chosen randomly between 0 and 1, then unit u_i is chosen for survey -I if,

$$\sum_{j=1}^{i-1} p_j^{(1)} < r \leq \sum_{j=1}^{i} p_j^{(1)}$$

and u_k is chosen for survey-II if,

$$\sum_{j=1}^{k-1} p_j^{(2)} < r \leq \sum_{j=1}^{k} p_j^{(2)}$$

It is expected that $|i-k|$ will be as small as possible, in a large number of cases, because of the serpentine arrangement of the frame. The probability restrictions are satisfied for both surveys.

Suppose two surveys, where the units are to be selected with two sets of initial probabilities $p_i^{(1)}$ and $p_i^{(2)}$ are to be integrated in the sense of ensuring a large number of common and adjacent units between the two samples. For this purpose, a map showing the location of all the units is secured and the units are numbered in serpentine manner such that neighbouring units receive consecutive serial numbers.

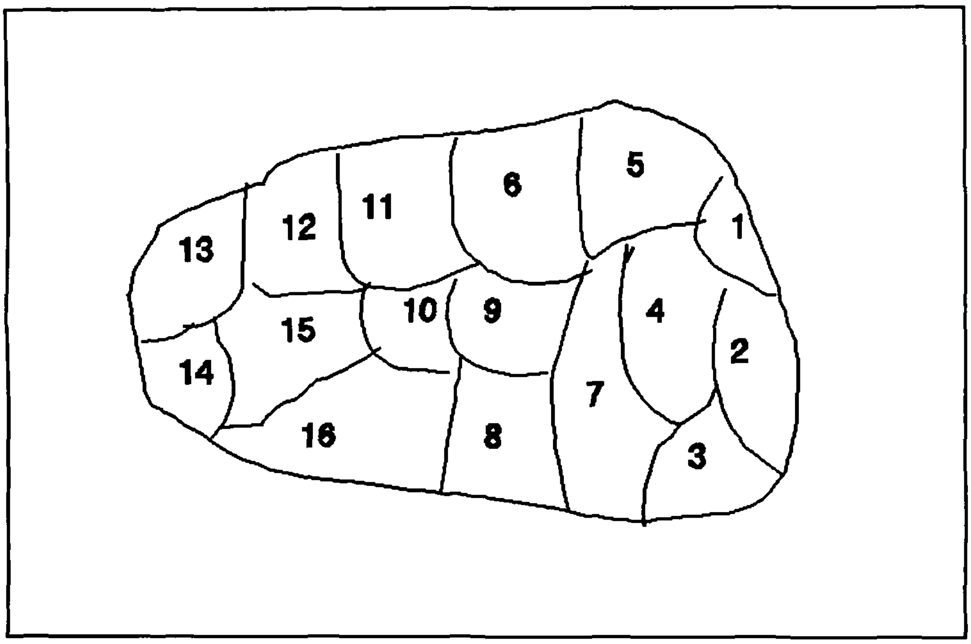

Fig. : 4.2.2 LAHIRIS SELECTION SCHEME OF SERPENTINE ORDERING.

4.2.3 <u>ROY CHOWDHURY'S SELECTION SCHEME</u> :

Roy Chowdhury (1956) offers a selection scheme as follows :

i) Select one unit for survey-I, say u_i, with probability $p_i^{(1)}$.

ii) Select one unit for survey-II other than u_i, say u_j, with probability,

$$p_j^{(2)} \Big/ \sum_{l \neq i} p_l^{(2)}$$

iii) Let $u_0 = (u_i, u_j)$. $u - \{u_i, u_j\}$ has N-2 elements with u_0 we have N-1 elements in all.

iv) Draw a sample of size n-1 from these N-1 units by simple random sampling without replacement.

v) If u_0 is selected in this sample of size (n-1), we select the unit u_j for survey-I and the unit u_i for survey-II. That is both u_i and u_j are in the samples for both the surveys and the remaining (n-2) units are also common. Otherwise, we have the set S of (n-1) distinct elements from u other than u_i and u_j. We have $\{u_i\}$ US for survey-I and $\{u_j\}$US for survey-II.

4.3 FORMULATION OF THE PROBLEM OF OPTIMUM INTEGRATION OF SURVEY AS A PROBLEM OF MATHEMATICAL PROGRAMMING :

Let P_{ij} be the probability with which unit u_i and u_j are selected for survey-I and survey-II respectively. Therefore if the unit u_i is included in a sample for survey-I with probability $p_i^{(1)}$ we must have,

$$\sum_{j=1}^{N} P_{ij} = p_i^{(1)}, \ i=1,.N \tag{4.3.1}$$

Similarly if the unit u_j is included in a sample for survey-II with probability $p_j^{(2)}$ we must have,

$$\sum_{i=1}^{N} P_{ij} = p_j^{(2)}, \ j=1.N \tag{4.3.2}$$

Furthermore as P_{ij}'s are probabilities we must have

$$0 \le P_{ij} \le 1, \ i, \ j = 1 \ ..... \ N \ .... \tag{4.3.3}$$

However, any set of nonnegative P_{ij} satisfying (4.3.2) and (4.3.3) for all i and j automatically satisfies the restrictions $P_{ij} \le 1$ as $p_i^{(1)}$ and $p_j^{(2)}$ are probabilities. Thus we may drop the restriction $P_{ij} \le 1$, i,j=1....n and we have the following problem.

Consider the following mathematical programming problem,

$$\text{Maximize} \sum_{j=1}^{N} P_{ii} \qquad \ldots . \quad (4.3.4)$$

$$\left.\begin{array}{l} \displaystyle\sum_{j=1}^{N} P_{ij} = p_i^{(1)} \quad i=1....N \\ \\ \displaystyle\sum_{i=1}^{N} p_{ij} = p_j^{(2)} \quad j=1....N \\ \\ p_{ij} \ge 0 \quad i,j=1,....N \end{array}\right] \qquad \ldots . \quad (4.3.5)$$

The objective function and the constraints of the above problem are linear functions and therefore the problem (4.3.4) is a linear programming problem. Furthermore the constraints may be expanded as,

$$\begin{array}{l} \\ \text{1st colu-} \\ \text{mn} \\ \\ \text{2nd} \\ \text{colu-} \\ \text{mn} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} \underline{\text{1st N columns}} & \underline{\text{2nd N columns}} & \underline{\text{Nth Columns}} \end{array} \\ \begin{pmatrix} 1 \;\; 1 \; \ldots\ldots \; 1 & 0 \; 0 \; \ldots\ldots\ldots 0 & 0 \; \ldots\ldots\ldots 0 \\ 0 \;\; 0 \; \ldots\ldots \; 0 & 1 \; 1 \; \ldots\ldots\ldots 1 & \ldots\ldots\ldots\ldots 0 \\ & & \\ 0 \;\; 0 \; \ldots\ldots \; 0 & 0 \; 0 \; \ldots\ldots\ldots 0 & 1 \; 1 \; \ldots\ldots 0 \\ 1 \;\; 0 \; \ldots\ldots \; 0 & 1 \; 0 \; \ldots\ldots\ldots 0 & 1 \; 0 \; \ldots\ldots 0 \\ 0 \;\; 1 \; \ldots\ldots \; 0 & 0 \; 1 \; \ldots\ldots\ldots 0 & 0 \; 1 \; \ldots\ldots 0 \\ & & \\ 0 \;\; 0 \; \ldots\ldots \; 1 & 0 \; \ldots\ldots\ldots\ldots 1 & 0 \; \ldots\ldots\ldots 1 \end{pmatrix} \end{array} \begin{pmatrix} P_{11} \\ P_{12} \\ P_{1N} \\ P_{21} \\ P_{2N} \\ P_{N1} \\ \\ P_{NN} \end{pmatrix} = \begin{pmatrix} p_1^{(1)} \\ p_2^{(2)} \\ \\ p_N^{(1)} \\ p_1^{(2)} \\ p_2^{(2)} \\ \\ p_N^{(2)} \end{pmatrix}$$

$$\text{or } AP = p \; \ldots\ldots \qquad (4.3.6)$$

where A is a ($2N \times N^2$) matrix, P is an N^2 component vector and p is a 2N component vector. A linear programming problem with constraint matrix of the type as described above is called a transportation problem. It can be seen that the rank of A is (2N-1) and consequently any basic feasible solution to the problem (4.3.4) - (4.3.5) can have at most (2N-1) nonzero P_{ij}'s.

For the problem (4.3.4) - (4.3.5) we have the following results.

<u>LEMMA</u> : In any optimal solution $P^* = ((P_{ij}))$ to the problem (4.3.4) - (4.3.5)

$$P_{ii}^* = \min (p_i^{(1)}, p_j^{(2)}) \text{ for all } i,j = 1 \ldots N$$

<u>PROOF</u> : In any feasible solution to problem (4.3.4) no P_{ii} can be greater than $\min (p_i^{(1)}, p_j^{(2)})$, as that would violate the constraints. Therefore in an optimal solution P^*, P_{ii}^* can be less than or equal to $\min (p_i^{(1)}, p_j^{(2)})$. Suppose for some k, $P_{kk}^* <$ $\min (p_k^{(1)}, p_k^{(2)})$. Then there exist $P_{ks}^* > 0$ and $P_{rk}^* > 0$ some s and r, $s \neq k \neq r$.

Let $\theta = \min (P_{ks}^*, P_{rk}^*)$. Then the solution P given by.

$$P_{ij} = P_{ij}^* \quad r \neq i \neq k$$

$$s \neq j \neq k \quad i,j = 1 \ldots\ldots N$$

$$P_{rk} = P_{rk}^* - \theta$$

$$P_{ks} = P_{ks}^* - \theta$$

$$P_{rs} = P_{rs}^* + \theta$$

and $$P_{kk} = P_{kk}^* + \theta$$

Now, P is feasible and the objective function value corresponding to P is at least,

$$\sum_{i=1}^{N} P_{ii}^* + \theta$$

As $\theta > 0$ this leads to a contradiction as P^* is assumed to be optimal. Hence the result.

This result implies that the optimal objective function value is given by Σ min $(p_i^{(1)}, p_j^{(2)})$. Hence we have the following result.

<u>LEMMA</u> : A necessary and sufficient condition for optimality of any feasible solution P is P_{ii} = min $(p_i^{(1)}, p_i^{(2)})$ for all i.

<u>NOTE</u> : In Keyfitz's scheme we select u_i for both the surveys, with the probability P_{ii} = min $(p_i^{(1)}, p_i^{(2)})$. Since if $(p_i^{(2)} \geq p_i^{(1)})$, we include u_i with probability 1(one) in the sample selected for survey-II. That is, $P_{ii} = p_i^{(1)}.1 = p_i^{(1)}$ = min $(p_i^{(2)}, p_i^{(1)})$. In case when $(p_i^{(2)} \leq p_i^{(1)})$, we include u_i with probability $(p_i^{(2)} / p_i^{(1)})$ in the sample selected for survey-II. That is $P_{ii} = (p_i^{(1)}, p_i^{(2)}) / (p_i^{(1)}$ = min $(p_i^{(2)}, p_i^{(1)})$. Also, for $i \neq j$ in Keyfitz's scheme we have,

$$P_{ij} = (p_i^{(1)} - p_i^{(2)})\ [p_j^{(2)} - p_j^{(1)}]/p_j^{(2)} - p_j^{(1)}$$

for $i \in I$ $j \in J$ ... (4.3.7)

$= 0 \qquad$ otherwise

where $J = \{j \mid (p_j^{(2)} - p_j^{(1)}) > 0\}$ and

$I = \{i \mid (p_i^{(1)} - p_i^{(2)}) > 0\}$

This shows that Keyfitz's scheme provides an optimal solution to problem (4.3.4) - (4.3.5).

4.3.1 EXAMPLE :

The following table shows the probabilities of selection of four villages in two surveys viz, the crop survey (survey-I) and demographic survey (survey-II).

	Villages 1	2	3	4
Survey-I $p_i^{(1)}$	0.5	0.2	0.1	0.2
Survey-II $p_i^{(2)}$	0.3	0.1	0.4	0.2

TABLE -4.3.1

Keyfitz's scheme provides the following P_{ij}'s for the problem is,

$P_{11} = \min(0.5, 0.3) = 0.3$

$P_{22} = \min(0.2, 0.1) = 0.1$

$P_{33} = \min(0.1, 0.4) = 0.1$

$P_{44} = \min(0.2, 0.2) = 0.2$

For calculating other P_{ij}'s the set of indices are :

$$J = \{ j \mid p_j^{(2)} - p_j^{(1)} > 0 \} = \{3\}$$

$$I = \{ i \mid p_j^{(1)} - p_j^{(2)} > 0 \} = \{1,2\}$$

(4.3.6) gives

$$P_{12} = 0$$

$$P_{13} = \frac{(0.5 - 0.3)(0.4 - 0.1)}{(0.4 - 0.1)} = 0.2$$

$$P_{14} = 0$$

$$P_{21} = 0$$

$$P_{23} = \frac{(0.2 - 0.1)\ (0.4 - 0.1)}{(0.4 - 0.1)} = 0.1$$

$P_{24} = 0$ $\qquad$ $P_{41} = 0$

$P_{31} = 0$ $\qquad$ $P_{42} = 0$

$P_{32} = 0$ $\qquad$ $P_{43} = 0$

$P_{34} = 0$

The value of P_{ij}'s arranged in a matrix form are,

i	1	2	3	4	$p_i^{(1)}$
1	0.3	0	0.2	0	0.5
2	0	0.1	0.1	0	0.2
3	0	0	0.1	0	0.1
4	0	0	0	0.2	0.2
p	0.3	0.1	0.4	0.2	1

TABLE - 4.3.2

The optimum value of the objective function is,

$\Sigma P_{ii} = 0.3 + 0.1 + 0.1 + 0.2 = 0.7$

according to Keyfitz's selection scheme.

4.4 ANOTHER MATHEMATICAL PROGRAMMING FORMULATION :

Consider the problem

Minimize $\Sigma\Sigma P_{ij} \mid j - i \mid$ ..... (4.4.1)

Subject to,

$$\sum_{j=1}^{N} P_{ij} = p_i^{(1)} \quad i=1..N$$

$$\sum_{i=1}^{N} p_{ij} = p_j^{(2)} \quad j=1,.N$$

$$P_{ij} \geq 0 \quad for \quad i,j=1,2..N$$

Problem (4.3.4) and (4.4.1) has the same feasible region. That is, the problem (4.4.1) - (4.4.2) is also a transportation problem. The coefficients $C_{ij} = |j-i|$ and $p_i^{(1)}$ and $p_j^{(2)}$ for i,j = 1, 2, 3, 4 are.

$$((C_{ij})) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

4.4.1 SELECTING A BASIC FEASIBLE SOLUTION :

A number of methods are available for selecting an initial basic feasible solution to a transportation problem. In the following we give the North-West Corner Rule (NWCR).

North West Corner Rule : Select the cell (r,s) such that (r+s) is a minimum and assign P_{rs} the maximum possible value. The steps involved in this method are :

Step 1 : The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the table. The maximum feasible amount is allocated there. That is $P_{11} = \min (p_1^{(1)}, p_1^{(2)})$.

Step 2 : If $p_1^{(2)} > p_1^{(1)}$, we move down vertically to the second row and make the second allocation as $P_{21} = \min (p_2^{(1)}, p_1^{(2)} - P_{11})$ in the cell (2,1)

If $p_1^{(2)} < p_1^{(1)}$, there is a tie for the second allocation, we can either allocate in the cell (1,2) or (2,1) with the magnitude discuss above.

Step 3 : Repeat steps 1 and 2 moving towards the lower right corner of the table until the rim requirements $p_i^{(1)}$ and $p_j^{(2)}$ are satisfied.

4.4.2 SOLUTION OF A TRANSPORTATION PROBLEM : THE MODIFIED DISTRIBUTION METHOD

The following are the steps involved in this method :

Step 1 : Determine an initial basic feasible solution by using NWCR. Enter the solution in the upper left corners of the basic cells.

Step 2 : For all the basic variables P_{ij}, solve the system of equations, $u_i + v_j = c_{ij}$ for all i and j which is in the basis. Starting initially with some $u_i = 0$ and compute the values u_i and v_j and display them along on the transportation table.

Step 3 : Compute the net evaluations $Z_{ij} - C_{ij} = u_i + v_j - C_{ij}$ for all the non-basic cells and enter them in the upper right corners of the corresponding cells.

Step 4 : If all $Z_{ij} - C_{ij} \leq 0$, then the current basic feasible solution is optimum. If at least one $Z_{ij} - C_{ij} > 0$, select the variables P_{rs} having the largest positive net evaluations to enter the basis.

Step 5 : Let the variable P_{rs} enter the basis. Allocate an unknown quantity, say θ, to the cell (r,s). Identify a loop, where loop is defined as a set of four or more ordered cell if :

i) any two adjacent cells in the ordered set lie either in the same row or in the same column, and
ii) any three or more adjacent cells do not lie in the same row or column.

Select a loop that starts and ends at the cell (r,s) and connect some of the basic cells. Add and subtract interchangeably θ to and from the transition cells of the loop in such a way that the rim requirements remain satisfied.

Step 6 : Assign a maximum value to θ in such a way that the value of one basic variable becomes zero and the other basic variable remain non-negative. The basic cell reduced to zero leaves the basis.

Step 7 : Return to step 3 and repeat the process until an optimum solution obtained.

4.4.3 DEGENERACY IN TRANSPORTATION PROBLEM :

If in any basic feasible solution to (4.4.1) - (4.4.2), the number of basic cell is less than (2N-1), the basic feasible solution is said to be a degenerate one.

To resolve degeneracy, we augment the positive variables by as many zero-valued variables as necessary to complete the required (2N-1) basic cells. These zero-valued variables are so chosen that the resulting (2N-1) variables constitute a basic solution. The zero-valued variables are designated by allocating an extremely small positive value ϵ. The cells containing these extremely small allocations are then treated like any other basic cell. The ϵ's are kept in the table until temporary degeneracy is removed or until the optimum solution is attained. At that point we set $\epsilon = 0$.

4.4.4 OPTIMALITY OF NORTH-WEST CORNER RULE :

LEMMA : A basic feasible solution obtained by NWCR is optimal for problem (4.4.1) - (4.4.2).

PROOF : Let P = ((Pij)) and C(P) denote the objective function value corresponding to the solution P. By North-West Corner rule we select one basic cell in each stage, until we have 2N-1 basic cells in all. Let the cell selected at the 1^{th} stage by designated as the 1^{th} basic cell. Let the adjusted values of $p_i^{(1)}$, $p_j^{(2)}$ at the 1^{th} stage be denoted by $p_i^{(1)}$ (1) and $p_j^{(2)}$ (1), respectively. Let P^* denote the matrix corresponding to the basic feasible solution.

Suppose P^* is not optimal. Let there exist a $P \neq P^*$ such that $C(P) < C(P^*)$. Now let l be the smallest positive integer such that the first (l-1) north-west corner cells of P^* agree with P. Let the next north-west corner cell determined be the cell (r,s), as the cell (r,s) is such that $P_{rs}^* = \min \{p_r^{(1)}$ (1), $p_s^{(2)}$ (1)}. If $P_{rs} \neq P_{rs}^*$ it must be $P_{rs} < P_{rs}^*$. Therefore there exist k,l such that $r < k$ and $s < l$ such that $P_{ks} > 0$ and $P_{rl} > 0$. Now from the result that, with the above condition, there exists $P' = ((P_{ij}'))$ such that $C(P') \leq C(P)$ and $P_{rs}' > P_{rs}$ (for proof see Arthanari and Dodge (1981). Thus we can show that there exists a P′ as good as P and $P'_{rs} > P_{rs}$.

If $P'_{rs} > P_{rs}^*$, we have shown, if there exists a P agreeing with P^* up to the first (l-1) NWC cells then there exists a P′ agreeing with P^* up to the first l NWC cells.

If $P'_{rs} < P_{rs}$, we can repeat with $P = P'$ till P_{rs}^* equal the new P_{rs} obtained. Hence the result.

4.4.5 AN EXAMPLE :

Using the same data as in example (4.3.1) an initial basic feasible to the problem (4.4.2) - (4.4.3) obtained by NWCR is

	1	2	3	4	$p_i^{(1)}$
1	$\boxed{.3}$ 0	$\boxed{.1}$ 1	$\boxed{.1}$ 2	3	.5
2	1	0	$\boxed{.2}$ 1	2	.2
3	2	1	$\boxed{.1}$ 0	1	.1
4	3	2	1	$\boxed{.2}$ 0	.2
$P_j^{(2)}$	0.3	0.1	0.4	0.2	1

The value of the objective function is,

$$Z = \Sigma\Sigma C_{ij} P_{ij}$$

$$= 0.5 \quad \text{when } C_{ij} = |\ j-i\ |.$$

4.5 <u>DISCUSSION</u> : In this chapter we discussed the various selection scheme considered for integration of surveys. In Lahiri's serpentine method of numbering units we find that two units geographically contiguous to a third unit may not be so considered, as they receive different numbers. In mathematical model considered by Raj (1956) the costs are proportional to $|j-i|$. In figure (4.2.2) we see that the unit u_5 and u_2 are adjacent to unit u_1 but we give a cost 4 for unit u_1 to u_5 and only 1 for unit u_1 to u_2. Instead we may use any C_{ij} that reflects the actual cost of travelling from u_i to u_j. In that case we have a general transportation problem. The problem of optimal integration of surveys constitutes a large transportation problem. Aragon and Pathak (1990) provides a method of finding an optimal solution for integrating two samples of the same size which reduces the given transportation problem to a new problem whose size is more than one fourth of the size of the original problem.

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